Day 11  Homework.

① Please calculate the following in the $\mathbb{Z}_n$ specified.

(1) $8 \oplus 10$ in $\mathbb{Z}_{10}$
(2) $8 \oplus 6$ in $\mathbb{Z}_{10}$
(3) $8 \oplus 7$ in $\mathbb{Z}_{10}$
(4) $5 \oplus 9$ in $\mathbb{Z}_{40}$
(5) $50 \oplus 21$ in $\mathbb{Z}_{70}$

(6) $1 \oplus 13$ in $\mathbb{Z}_{100}$. 
(7) \( 40 \equiv 31 \mod 43 \)

6. Let \( n \) be a positive integer, and let \( a, b \in \mathbb{Z}_n \) be invertible. Prove or disprove each of the following statements.

1. \( a \oplus b \) is invertible.

2. \( a \odot b \) is invertible.

3. \( a \otimes b \) is invertible.

4. \( a \boxdot b \) is invertible.
Suppose $n \geq 2$. Show that $n^{-1}$ is invertible in $\mathbb{Z}_n$ and that it is its own inverse, i.e., 
\[(n^{-1})^{-1} = n^{-1}.
\]

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ be regular (not modular) matrices.

1. Compute $AB$.

2. Compute $\det(AB)$ (i.e., the determinant of $AB$).

3. Compute $\det(A)$ and $\det(B)$.

4. Is $\det(A)\det(B)$ equal to $\det(AB)$?
(2) Consider a matrix \( A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \).

(1) Find a constant \( c \) so that
\[
\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = c \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

(2) The constant \( c \) you found in (1) is called an eigenvalue of \( A \), and the vector \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) is called the eigenvector of \( A \) associated with the eigenvalue \( c \).

Find another eigenvalue and the corresponding eigenvector \( \begin{pmatrix} x \\ y \end{pmatrix} \) such that the eigenvector is not a constant multiple of \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \), i.e., \( \begin{pmatrix} x \\ y \end{pmatrix} \neq \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) for any constant \( \alpha \). (Note: the zero vector \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) is not considered an eigenvector.)