

3. Let $f \in C^n[a, b]$. Prove that if $x_0 \in (a, b)$ and if x_1, x_2, \dots, x_n all converge to x_0 , then $f[x_0, x_1, \dots, x_n]$ will converge to $f^{(n)}(x_0)/n!$.

4. Prove that if f is a polynomial of degree k , then for $n > k$,

$$f[x_0, x_1, \dots, x_n] = 0$$

5. Prove that if p is a polynomial of degree at most n , then

$$p(x) = \sum_{i=0}^n p[x_0, x_1, \dots, x_i] \prod_{j=0}^{i-1} (x - x_j)$$

6. Show that the divided differences are linear maps on functions. That is, prove the equation

$$(\alpha f + \beta g)[x_0, x_1, \dots, x_n] = \alpha f[x_0, x_1, \dots, x_n] + \beta g[x_0, x_1, \dots, x_n]$$

7. The divided difference $f[x_0, x_1]$ is analogous to a first derivative, as indicated in Theorem 4. Does it have a property analogous to $(fg)' = f'g + fg'$?

8. Using the functions ℓ_i defined in Section 6.1 (p. 312) and based on nodes x_0, x_1, \dots, x_n , show that for any f ,

$$\sum_{i=0}^n f(x_i) \ell_i(x) = \sum_{i=0}^n f[x_0, x_1, \dots, x_i] \prod_{j=0}^{i-1} (x - x_j)$$

9. (Continuation) Prove this formula:

$$f[x_0, x_1, \dots, x_n] = \sum_{i=0}^n f(x_i) \prod_{\substack{j=0 \\ j \neq i}}^n (x_i - x_j)^{-1}$$

10. Compare the efficiency of the divided difference algorithm to the procedure described in Section 6.1 (p. 311) for computing the coefficients in a Newton interpolating polynomial.

11. Use Cramer's rule in matrix theory to prove that

$$f[x_0, x_1, \dots, x_n] = \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^{n-1} & f(x_0) \\ 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} & f(x_1) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} & f(x_n) \end{vmatrix} \div \begin{vmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix}$$

12. For the particular function $f(x) = x^m$, $m \in \mathbb{N}$, show that

$$f[x_0, x_1, \dots, x_n] = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n > m \end{cases}$$

13. Prove the Leibniz formula

$$(fg)[x_0, x_1, \dots, x_n] = \sum_{k=0}^n f[x_0, x_1, \dots, x_k] g[x_k, x_{k+1}, \dots, x_n]$$

Hint: See Problem 6.2.7 (above).

14. Write the equation in Problem 6.2.9 (above) in this form:

$$f[x_0, x_1, \dots, x_n] = \sum_{i=0}^n \alpha_i f(x_i) \quad \text{where} \quad \alpha_i = \prod_{\substack{j=0 \\ j \neq i}}^n (x_i - x_j)^{-1}$$

given by

$$p(x) = \sum_{k=0}^m \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k$$

The coefficient of x^m in $p(x)$ is $f^{(m)}(x_0)/m!$, and this establishes Equation (1) when $n = m$ and $x_m = x_0$. In the other case, the argument used to prove Theorem 1 in Section 6.2 (p. 330) is valid.

PROBLEMS 6.3

1. Use the extended Newton divided difference method to obtain a quartic polynomial that takes these values:

x	0	1	2
$p(x)$	2	-4	44
$p'(x)$	-9	4	

2. (Continuation) Find a quintic polynomial that takes the values given in the preceding problem and, in addition, satisfies $p(3) = 2$. *Hint:* Add a suitable term to the polynomial found in the preceding problem.

3. Obtain a formula for the polynomial p of least degree that takes these values:

$$p(x_i) = y_i \quad p'(x_i) = 0 \quad (0 \leq i \leq n)$$

4. What condition will have to be placed on the nodes x_0 and x_1 if the interpolation problem

$$p(x_i) = c_{i0} \quad p''(x_i) = c_{i2} \quad (i = 0, 1)$$

is to be solvable by a cubic polynomial (for arbitrary c_{ij})?

5. Show that the polynomial given in Equation (6), with the divided differences in Equations (4) and (5), fulfills the conditions in Equation (3).

6. Verify the properties claimed for the functions A_i and B_i .

7. Prove that the Taylor polynomial

$$p(x) = \sum_{j=0}^{k-1} \frac{1}{j!} f^{(j)}(x_0)(x - x_0)^j$$

interpolates f at x_0, x_0, \dots, x_0 (k repetitions).

8. Prove that a polynomial interpolates 0 at x_0, x_1, \dots, x_n (repetitions permitted) if and only if it contains the factor $\prod_{j=0}^n (x - x_j)$.

9. Prove that if f interpolates g at x_0, x_1, \dots, x_n and if h interpolates 0 at these points, then $f \pm ch$ interpolates g at these points.

10. Fix x_0, x_1, \dots, x_n and show that the set of functions that interpolate 0 at these points is an algebra; that is, it is closed under addition, multiplication, and multiplication by scalars.

11. Prove that if f interpolates 0 at the nodes x_0, x_1, \dots, x_n , then it interpolates 0 at the nodes x_0, x_1, \dots, x_{n-1} .

12. Let $x_0 < x_1 < \dots < x_n$, and let f be continuously differentiable. Show that

$$\frac{\partial}{\partial x_i} f[x_0, x_1, \dots, x_n] = f[x_0, x_1, \dots, x_i, x_i, x_{i+1}, \dots, x_n]$$