\[= \sum_{i=1}^{m} (P^*)_{ki} \sum_{j=1}^{n} A_{ij} \bar{e}_j = \sum_{i=1}^{m} (P^*)_{ki} \sum_{j=1}^{n} A_{ij} \bar{e}_j\]
\[= \sum_{j=1}^{n} (P^*A)_{kj} \bar{e}_j = \sum_{j=1}^{n} (P^*A)_{kj} \sum_{s=1}^{n} (\bar{e}_j, v_s) v_s\]
\[= \sum_{j=1}^{n} \sum_{s=1}^{n} (P^*A)_{kj} (Q^*)_j s v_s = \sum_{s=1}^{n} (P^*AQ)^* s v_s\]

Now the equation \(A = PDQ\) leads to \(P^*AQ = D\). The matrix \(D\) is a diagonal \(m \times n\) matrix having the singular values of \(A\), \(\sigma_1, \sigma_2, \ldots, \sigma_\ell\) on its diagonal, with \(\ell = \min(m, n)\). Hence,

\[Lu_k = \sum_{j=1}^{n} D_{kj} v_j = \begin{cases} \sigma_k v_k & \text{if } k \leq \ell \\ 0 & \text{if } k > \ell \end{cases}\]

PROBLEMS 5.4

1. Find the singular-value decompositions of these matrices:

\[
\begin{bmatrix}
4 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 7 \\
0 & 0 & 0
\end{bmatrix} \quad \begin{bmatrix}
2 & 1 \\
\end{bmatrix} \quad \begin{bmatrix}
5 \\
\end{bmatrix}
\]

2. Find the minimal solutions of these systems of equations:

a. \(x_1 + x_2 = b_1\)

\[\begin{align*}
x_1 &= b_1 \\
x_2 &= \text{any value}
\end{align*}\]

b. \(x_1 = b_2\)

\[\begin{align*}
x_1 &= b_2 \\
x_2 &= \text{any value}
\end{align*}\]

c. \(4x_1 = b_1, \quad 0x_1 = b_2, \quad 7x_3 = b_3, \quad 0x_2 = b_4\)

3. Find \(A^+\) in the case that \(AA^*\) is invertible.

4. Find \(A^+\) in the case that \(A^*A = I\).

5. Find \(A^+\) in the case that \(A^*A = A\) and \(A^2 = A\).

6. Prove that if \(A\) is Hermitian, then so is \(A^+\).

7. If \(A\) is Hermitian and positive definite, what is its singular-value decomposition?

8. Prove that the pseudoinverse is a discontinuous function of a matrix. *Hint: Compute the pseudoinverse of the matrix*
11. Prove these properties of the pseudoinverse:
   a. \((AA^+)^+ = A^+A^+\)
   b. \(A^+ = A^+(AA^+)^+\)

12. Show by a suitable example that in general \((AB)^+ \neq B^+A^+\).


14. Refer to the proof of Theorem 1 and prove that
   \[ A = \sum_{j=1}^{r} \sigma_j u_j v_j^* \]

(Compare with the preceding problem.)

15. (Continuation) Let \(A\) be an \(m \times n\) matrix of rank \(r\). Define \(u_1, v_1, \ldots, u_r, v_r\) as in the proof of Theorem 1. Show that
   \[ A^+ = \sum_{j=1}^{r} \sigma_j^{-1} u_j v_j^* \]

(Compare with the preceding problem.)

16. (Continuation) Use the result of Problem 5.4.14 above to show that if the singular-value decomposition of \(A\) is available, then \(AX\) can be computed at a cost of \((n + m + 1)r\) multiplications and \(r( n + m - 1) - m\) additions. Compare this to the straightforward multiplication, which costs \(nm\) multiplications and \((n - 1)m\) additions.

17. (Continuation) If the sum in Problem 5.4.14 above is truncated with \(k\) summations, we obtain an approximation to \(A\). Prove that the error in doing so satisfies
   \[ \left\| A - \sum_{j=1}^{k} \sigma_j u_j v_j^* \right\|_2 = \sigma_{k+1} \]

where the matrix norm subordinate to the Euclidean vector norm has been used.

18. Let \(A\) be an \(m \times n\) matrix of rank \(r\), with \(m \geq n \geq r\) and singular-value decomposition \(A = PD\). Prove that the system of equations \(Ax = b\) is consistent if and only if \((P^Tb)_i = 0\) for \(r < i \leq m\).

19. Prove that if \(A\) is Hermitian and positive semidefinite, then its eigenvalues are identical with its singular values.

20. Prove that if two matrices are unitarily equivalent, then their singular values are the same.

(Two matrices \(A\) and \(B\) are unitarily equivalent if \(A = UBV\) for suitable unitary matrices \(U\) and \(V\).)

21. Let \(A\) be an \(n \times n\) matrix having singular values \(\sigma_1, \sigma_2, \ldots, \sigma_n\). Prove that the determinant of \(A\) is \(\det(A) = \pm \sigma_1 \sigma_2 \cdots \sigma_n\).

22. Let \(\|A\|_2\) denote the matrix norm subordinate to the Euclidean vector norm. Let the singular values of \(A\) be \(\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n\). Show that \(\|A\|_2 = \sigma_1\).

23. Let \(A\) be a square matrix with singular-value decomposition \(A = PD\). Prove that the characteristic polynomial of \(A\) is \(\pm \det(D - \lambda P^TQ) = 0\).

24. Let \(A\) be an \(m \times n\) matrix, and let \(X\) be an \(n \times m\) matrix that has the four Penrose properties relative to \(A\). Prove that the minimal solution to the system \(AX = b\) is \(Xb\).

25. Prove that if \(A\) is real, then it has a real singular-value decomposition and a real pseudoinverse.
26. Prove that the squares of the elements in \( w_i u_i^* \) sum to 1. (Notation is as in the proof of Theorem 1.)

27. Suppose that \( A = UDV \), where \( U \) is \( m \times m \) unitary, \( V \) is \( n \times n \) unitary, and \( D \) is \( m \times n \) diagonal. Show that \( |d_{ii}|^2 (1 \leq i \leq n) \) are the eigenvalues of \( A^*A \).

28. Prove the three remaining Penrose properties for \( D^+ \) (see Theorem 4).

29. (Continuation) Complete the proof of Theorem 4.

30. Prove that if the \( m \times n \) matrix \( A \) has rank \( n \), then \( A^+ = (A^*A)^{-1}A^* \).

31. Prove that the pseudoinverse of a diagonal \( m \times n \) matrix is a diagonal \( n \times m \) matrix.

32. Find the pseudoinverse of an arbitrary \( m \times 1 \) matrix and of an arbitrary \( 1 \times n \) matrix.

33. Show that the orthogonal projection of \( C^n \) onto the column space of \( A \) is \( AA^+ \).

34. Prove that if \( A \) is symmetric, then so is \( A^+ \).

35. Prove that the equation \((AB)^+ = B^+A^+\) holds if \( A \) and \( B \) are of full rank. An \( m \times n \) matrix \( A \) is said to have full rank if \( \text{rank}(A) = \min(m, n) \).

36. Find the pseudoinverse of \( uv^* \), where \( u \) and \( v \) are members of \( C^n \).

37. Find the pseudoinverse of an \( m \times n \) matrix, all of whose elements are 1's.

38. Use Theorem 3 to prove that if \( B \) is an \( m \times r \) matrix, if \( C \) is an \( n \times r \) matrix, and if \( B, C \) and \( BC \) all have rank \( r \), then \( (BC)^+ = C^*(C^+)^{-1}(B^*B)^{-1}B^* \).

39. Prove that the eigenvalues of a positive semidefinite matrix are nonnegative.

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**COMPUTER PROBLEM 5.4**

1. Write a computer program to find the minimal solution of a system of equations, \( Ax = b \), using the singular-value decomposition and pseudoinverse.

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### 5.5 QR-Algorithm of Francis for the Eigenvalue Problem

In Section 5.2 (p. 266), we proved Schur's Theorem, according to which any square matrix is unitarily similar to a triangular matrix. Thus, a factorization of the type

\[
UAU^* = T
\]

is possible, where \( U \) is unitary and \( T \) is triangular. Since the eigenvalues of \( A \) and \( T \) are the same, and since the eigenvalues of a triangular matrix are simply its diagonal elements, we shall find the eigenvalues of \( A \) displayed on the diagonal of \( T \).

Although, we know that the factorization in Equation (1) exists for any given matrix \( A \), it is not a simple matter to compute it. Finding \( U \) must be as difficult as finding

---

**QR-Factorization of a Matrix**

The QR-algorithm for finding the eigenvalues of a matrix is based on the algorithm introduced in Section 4.3.4.

In Section 4.3.4, we defined a certain matrix \( A_k \) using the following formula:

\[
A_k = A_{k-1} - \frac{r_{kk}}{r_{kk}^2 + \varepsilon} \lambda_k \begin{pmatrix} \frac{r_{kk}}{r_{kk}^2 + \varepsilon} & \frac{\varepsilon}{r_{kk}^2 + \varepsilon} \\ \frac{\varepsilon}{r_{kk}^2 + \varepsilon} & \frac{1}{r_{kk}^2 + \varepsilon} \end{pmatrix}
\]

where \( Q \) is unitary, \( R \) is upper triangular, and \( \lambda_k \) is the eigenvalue of \( A_k \) that we are finding.

In fact, if Equations (1) and (2) are used, it is easy to show that

\[
A_k = AQ_kA^*
\]

The definition of \( Q_k \) is:

\[
Q_k = \begin{pmatrix} I_k & T_k \end{pmatrix}
\]

where \( I_k \) is the \( k \times k \) identity matrix and \( T_k \) is the upper triangular matrix.

The QR-algorithm is then defined by the recursion:

\[
A_1 \leftarrow A \\
\text{for } k = 1 \text{ to } n \\
\text{QR-factorize } A_k \\
R_k \leftarrow \text{find upper triangular factor } T_k \\
A_{k+1} \leftarrow (I_k - \frac{T_k}{r_{kk}^2 + \varepsilon}) A_k
\]

end do

If the circulant matrix \( R_k \) is reduced to \( A_k \), then \( A_{k+1} \) is a circulant matrix whose eigenvalues are the same as those of \( A_k \).

The algorithm also reduces the size of the matrix, making it easier to store and process.

In practice, the algorithm is applied to a sequence of smaller matrices that are derived from the original matrix by removing rows and columns. The eigenvalues of these smaller matrices are then used to approximate the eigenvalues of the original matrix.

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**Second, we discuss the block algorithm for finding the eigenvalues of a matrix**