21. Let $A$ be an $m \times n$ matrix of rank $r$. Let $p$ be any point in $\mathbb{R}^n$. Show that the set

\[ \text{null}(A) = \{ x \in \mathbb{R}^n | Ax = 0 \} \]

forms a subspace of $\mathbb{R}^n$. Explain why this is a basis for $\text{null}(A)$.

22. Show that in solving the least-squares problem for the equation $Ax = b$, we can represent

\[ \min_{x \in \mathbb{R}^n} \| b - Ax \| \]

as

\[ \min_{x \in \mathbb{R}^n} \| Ax - b \| \]

23. Let $A$ be a diagonal matrix and $B$ a unitary matrix. Under what conditions is $A + B$ a unitary matrix?

24. Let $A$ be a diagonal matrix and $B$ a unitary matrix. Under what conditions is $A + B$ a unitary matrix?

25. Prove that when $x$ is as defined

\[ q = x \in \mathbb{R}^n \]

Prove that $x$ is a minimum when $x$ is a solution of the equation

\[ \| x \| = \min_{y \in \mathbb{R}^n} \| y \| \]

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Theorem 1

The existence of the singular-value decomposition (SVD) of an arbitrary complex matrix $A$ is guaranteed.

Proof

The matrix $A$ can be decomposed into $A = UDV^*$, where $U$ and $V$ are unitary matrices (and thus have singular values of 1 on the diagonal) and $D$ is a diagonal matrix with non-negative real entries (the singular values of $A$).

We begin our discussion with the singular-value decomposition.

5.4 Singular-Value Decomposition

A problem and its solution in the context of preparing for the exam.

The following example illustrates this point.

3. (Continuation) If $x^*$ is the least-squares solution of the equation $Ax = b$, how can the least-squares solution be viewed without solving for $x$?

$$x = \left[ \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right].$$

3.1. Program both the Gram-Schmidt and orthogonal projection methods.

1. Problem 5.3

39. Apply the Gram-Schmidt

30. Determine $x^*$ and $Ax^*$.

3.2. Let $A$ be an $m \times n$ matrix of unspecified rank. Let $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.

26. Let $A$ be an $m \times n$ matrix of unspecified rank. Let $x \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$.

27. (Continuation) If $x^*$ is the least-squares solution of the equation $Ax = b$, how can the least-squares solution be viewed without solving for $x$?

28. Prove that the inner product is additive in one vector and linear in the other vector.

29. Prove that the inner product is bilinear in both vectors.

2.9. Let $x$ be a vector in $\mathbb{R}^n$ and $y$ be another vector in $\mathbb{R}^n$.

1. Program both the Gram-Schmidt and orthogonal projection methods.