

Proof If it is possible to write $0 = \sum_{i=0}^n \theta_i u_i$ with $\theta_i \geq 0$, $\sum_{i=0}^n \theta_i = 1$, and $\min_i \theta_i = 0$, then do so and select k so that $\theta_k = 0$. Clearly we can replace u_k by u_{n+1} in the equation $0 = \sum_{i=0}^n \theta_i u_i$.

Suppose that it is *not* possible to write $0 = \sum_{i=0}^n \theta_i u_i$ with $\theta_i \geq 0$, $\sum_{i=0}^n \theta_i = 1$, and $\min_i \theta_i = 0$. By Carathéodory's Theorem, the vectors u_0, u_1, \dots, u_n must span an n -dimensional space, which of course must be \mathbb{R}^n . Hence, we can find $\lambda_0, \dots, \lambda_n$ such that $u_{n+1} = \sum_{i=0}^n \lambda_i u_i$. Also, there exist $\theta_i > 0$ such that $0 = \sum_{i=0}^n \theta_i u_i$. Select k so that $\lambda_k/\theta_k \geq \lambda_i/\theta_i$ for all i . For $0 \leq i \leq n$, put $\theta'_i = \lambda_k \theta_i - \lambda_i \theta_k$. Also, let $\theta'_{n+1} = \theta_k$. Observe that $\theta'_k = 0$. Now we have

$$\begin{aligned} \sum_{i=0}^{n+1} \theta'_i u_i &= \sum_{i=0}^n \theta'_i u_i + \theta'_{n+1} u_{n+1} = \sum_{i=0}^n (\lambda_k \theta_i - \lambda_i \theta_k) u_i + \theta_k u_{n+1} \\ &= \lambda_k \sum_{i=0}^n \theta_i u_i - \theta_k \sum_{i=0}^n \lambda_i u_i + \theta_k u_{n+1} \\ &= \lambda_k (-\theta_k u_k) - \theta_k (u_{n+1} - \lambda_k u_k) + \theta_k u_{n+1} = 0 \end{aligned}$$

Furthermore, $\theta'_i \geq 0$ because $\theta'_{n+1} = \theta_k > 0$, and for $i \leq n$,

$$\theta'_i = \theta_i \theta_k (\lambda_k / \theta_k - \lambda_i / \theta_i) \geq 0$$

If we divide the equation $\sum_{i=0}^{n+1} \theta'_i u_i = 0$ by $\sum_{i=0}^{n+1} \theta'_i$, we see that 0 is expressed as a convex combination of u_0, \dots, u_{n+1} , with u_k omitted. ■

PROBLEMS 6.9

- Solve for c_1, c_2, δ , and ξ in Example 1 of this section.
- Find the best approximation of \sqrt{x} by a first-degree polynomial on the interval $[0, 1]$.
- Show that the subspaces in $C[0, 1]$ spanned by these sets are Haar subspaces:
 - $\{1, x^2, x^3\}$
 - $\{1, e^x, e^{2x}\}$
 - $\{(x+2)^{-1}, (x+3)^{-1}, (x+4)^{-1}\}$
- Show that the subspaces in $C[-1, 1]$ spanned by these sets are not Haar subspaces:
 - $\{1, x^2, x^3\}$
 - $\{|x|, |x-1|\}$
 - $\{e^x, x+1\}$
- In the space $C[0, 1]$, consider the subspace Π_0 of polynomials of degree 0 (i.e., constant functions). Using the quantities

$$M(f) = \max_{0 \leq x \leq 1} f(x) \quad \text{and} \quad m(f) = \min_{0 \leq x \leq 1} f(x)$$

describe the best approximation of an element f in $C[0, 1]$ by an element of Π_0 .

- Let u_0, u_1, \dots, u_n be $n+1$ points in \mathbb{R}^n such that it is not possible to write $0 = \sum_{i=0}^n \theta_i u_i$ with $\theta \geq 0$, $\theta \neq 0$, and $\min_i \theta_i = 0$. Show that each set of n vectors chosen from $\{u_0, u_1, \dots, u_n\}$ is a basis for \mathbb{R}^n or give a counterexample.
- Let A be an $n \times (n+2)$ matrix. Show that if the system

$$Ax = 0 \quad x \geq 0 \quad x \neq 0 \quad x_{n+2} = 0 \quad x \in \mathbb{R}^{n+2}$$

is consistent, then for some $k < n + 2$, so is the system

$$Ax = 0 \quad x \geq 0 \quad x \neq 0 \quad x_k = 0 \quad x \in \mathbb{R}^{n+2}$$

8. Prove that the quadratic polynomial of best approximation to the function $\cosh x$ on the interval $[-1, 1]$ is $a + bx^2$, where $b = \cosh 1 - 1$ and a is obtained by solving this pair of equations simultaneously for a and t :

$$2a = 1 + \cosh t - t^2 b$$

$$\sinh t = 2tb$$

9. Prove that the convex hull of a set is convex and that it is the smallest convex set containing the original set.
10. Prove that in a normed linear space every closed ball $\{f : \|f - g\| \leq r\}$ is convex.
11. Give an example of a convex set whose complement is bounded.
12. Given a line in the plane, $ax + by + c = 0$, with $a^2 + b^2 > 0$, provide a complete description of the set of points on the line whose distance from the origin is a minimum, using the ℓ_∞ -norm to define distance.
13. Let f be a continuous function of (x, y) in the square $0 \leq x \leq 1, 0 \leq y \leq 1$. Describe the best approximation of f by a continuous function of x alone.
14. Do the three functions

$$g_0(x, y) = 1 \quad g_1(x, y) = x \quad g_2(x, y) = y$$

generate a Haar subspace in $C(\mathbb{R}^2)$?

15. Prove that the set of functions $\{1, x, x^2, \dots, x^{n-1}, f\}$ generates a Haar subspace on $[a, b]$ if $f^{(n)}(x) > 0$ on $[a, b]$.
16. Let $f = a_0 T_0 + a_1 T_1 + \dots + a_{n+1} T_{n+1}$, where the T_k are Chebyshev polynomials. Prove that the best approximation of f in the sup-norm on $[-1, 1]$ in the space Π_n is $a_0 T_0 + a_1 T_1 + \dots + a_n T_n$.

6.10 Interpolation in Higher Dimensions

The problem of finding smooth interpolants for functions of several variables is a difficult one that has attracted much attention, both in the past and currently. The multivariate case shows some unusual features that are not present in the univariate case, and these features are already apparent when the number of variables is only two. Therefore, very little is lost in restricting the discussion to the *bivariate* case (two independent variables), at least at the beginning.

Interpolation Problem

The central problem that we shall discuss is as follows: A set of interpolation points (or *nodes*) is given in the xy -plane. These can be denoted by

$$(x_1, y_1) \quad (x_2, y_2) \quad \dots \quad (x_n, y_n) \quad (1)$$

We assume that these n points are *distinct*. With each point (x_i, y_i) there is associated a real number, c_i , and our objective is to find a smooth and easily computed function