

```

input  $n, (a_{ij})$ 
for  $k = 1$  to  $n$  do
     $\ell_{kk} \leftarrow \left( a_{kk} - \sum_{s=1}^{k-1} \ell_{ks}^2 \right)^{1/2}$ 
    for  $i = k + 1$  to  $n$  do
         $\ell_{ik} \leftarrow \left( a_{ik} - \sum_{s=1}^{k-1} \ell_{is} \ell_{ks} \right) / \ell_{kk}$ 
    end do
end do
output  $(\ell_{ij})$ 

```

Theorem 2 guarantees that $\ell_{kk} > 0$. Observe that Equation (9) gives us the following bound for $j \leq k$:

$$a_{kk} = \sum_{s=1}^k \ell_{ks}^2 \geq \ell_{kj}^2$$

from which we conclude that

$$|\ell_{kj}| \leq \sqrt{a_{kk}} \quad (1 \leq j \leq k)$$

Hence, any element of L is bounded by the square root of a corresponding diagonal element in A . This implies that the elements of L do not become large relative to A even without any pivoting. (Pivoting is explained in the next section.)

In both the Cholesky and Doolittle algorithms, the dot products of vectors should be computed in double precision to avoid a buildup of roundoff errors. (See Computer Problem 2.2.6, p. 62–63.)

PROBLEMS 4.2

1. Prove these facts, needed in the proof of Theorem 2.
 - a. If U is upper triangular and invertible, then U^{-1} is upper triangular.
 - b. The inverse of a unit lower triangular matrix is unit lower triangular.
 - c. The product of two upper (lower) triangular matrices is upper (lower) triangular.
2. Prove that if a nonsingular matrix A has an LU -factorization in which L is a unit lower triangular matrix, then L and U are unique.
3. Prove that the forward substitution and back substitution algorithms and their permuted versions always solve $Ax = b$ if A is nonsingular.
4. (Continuation) Count the number of arithmetic operations involved in these four algorithms.
5. Prove that an upper or lower triangular matrix is nonsingular if and only if its diagonal elements are all different from 0.
6. Show that if all the principal minors of A are nonsingular and $\ell_{ii} \neq 0$ for each i , then $u_{kk} \neq 0$ for $1 \leq k \leq n$.

7. Prove that the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ does not have an LU -factorization. *Caution:* This is not a simple consequence of Theorem 1 proved in this section.
8. a. Write the **row version** of the **Doolittle algorithm** that computes the k th row of L and the k th row of U at the k th step. (Consequently, at the k th step, the order of computing is $\ell_{k1}, \ell_{k2}, \dots, \ell_{k,k-1}, u_{kk}, \dots, u_{kn}$.)
- b. Write the **column version** of the **Doolittle algorithm**, which computes the k th column of U and the k th column of L at the k th step. (Consequently, the order of computing is $u_{1k}, u_{2k}, \dots, u_{kk}, \ell_{k+1,k}, \dots, \ell_{nk}$ at the k th step.)
9. By use of the equation $UU^{-1} = I$, obtain an algorithm for finding the inverse of an upper triangular matrix. Assume that U^{-1} exists; that is, the diagonal elements of U are all nonzero.
10. A matrix $A = (a_{ij})$ in which $a_{ij} = 0$ when $j > i$ or $j < i - 1$ is called a **Stieltjes matrix**. Devise an efficient algorithm for inverting such a matrix.
11. Let A be an $n \times n$ matrix. Let (p_1, p_2, \dots, p_n) be a permutation of $(1, 2, \dots, n)$ such that (for $i = 1, 2, \dots, n$) row i in A contains nonzero elements only in columns p_1, p_2, \dots, p_i . Write an algorithm to solve $Ax = b$.
12. Show that every matrix of the form $A = \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$ has an LU -factorization. Show that even if L is *unit* lower triangular, the factorization is not unique. (This problem, as well as the next two problems, illustrate Taussky's Maxim: If a conjecture about matrices is false, it can usually be disproved with a 2×2 example.)
13. (Continuation) Show that every matrix of the form $A = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$ has an LU -factorization. Does it have an LU -factorization in which L is a *unit* lower triangular?
14. (Continuation) Show that every matrix of the following form has an LU -factorization. Does it have an LU -factorization in which L is a *unit* lower triangular?
- a. $A = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$
- b. $A = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$
15. Prove that if A is invertible and has an LU -decomposition, then all principal minors of A are nonsingular.
16. Let the system $Ax = b$ have the following property: There are two permutations of $(1, 2, \dots, n)$ called $p = (p_1, p_2, \dots, p_n)$ and $q = (q_1, q_2, \dots, q_n)$ such that, for each i , equation number p_i contains only the variables $x_{q_1}, x_{q_2}, \dots, x_{q_i}$. Write an efficient algorithm to solve this system.
17. Count the number of multiplications and/or divisions needed to invert a unit lower triangular matrix.
18. Prove or disprove: If A has an LU -factorization in which L is *unit* lower triangular, then it has an LU -factorization in which U is *unit* upper triangular.
19. Assuming that its LU -factorization is known, give an algorithm for inverting A . (Use Problem 4.2.9 above and Computer Problem 4.2.1, p. 163.)
20. Develop an algorithm for inverting a matrix A that has the property $a_{ij} = 0$, if $i + j \leq n$.
21. Use the Cholesky Theorem to prove that these two properties of a symmetric matrix A are equivalent.

- a. A is positive definite.
 b. There exists a linearly independent set of vectors $x^{(1)}, x^{(2)}, \dots, x^{(n)}$ in \mathbb{R}^n such that $A_{ij} = (x^{(i)})^T (x^{(j)})$.

22. Establish the correctness of the following algorithm for solving $Ux = b$ in the case that U is upper triangular:

```

for j = n to 1 step -1 do
  x_j ← b_j / u_jj
  for i = 1 to j - 1 do
    b_i ← b_i - u_ij x_j
  end do
end do

```

23. Prove that if all the leading principal minors of A are nonsingular, then A has a factorization LDU in which L is unit lower triangular, U is unit upper triangular, and D is diagonal.
24. (Continuation) Prove that if A is a symmetric matrix whose leading principal minors are nonsingular, then A has a factorization LDL^T in which L is unit lower triangular and D is diagonal.
25. (Continuation) Write an algorithm to compute the LDL^T -factorization of a symmetric matrix A . Your algorithm should do approximately half as much work as the standard Gaussian algorithm. *Note:* This algorithm can fail if some principal minors of A are singular. (This modification of the Cholesky algorithm does not involve square root calculations.)
26. Prove: A is positive definite and B is nonsingular if and only if BAB^T is positive definite.
27. If A is positive definite, does it follow that A^{-1} is also positive definite?
28. Consider

$$A = \begin{bmatrix} 2 & 6 & -4 \\ 6 & 17 & -17 \\ -4 & -17 & -20 \end{bmatrix}$$

Determine *directly* the factorization $A = LDL^T$, where D is diagonal and L is unit lower triangular; that is, do *not* use Gaussian elimination.

29. Develop an algorithm for finding directly the UL -factorization of a matrix A , where L is unit lower triangular and U is upper triangular. Give an algorithm for solving $ULx = b$.
30. Find the LU -factorization of the matrix

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & -1 & 3 \\ 1 & 3 & 0 \end{bmatrix}$$

in which L is lower triangular and U is unit upper triangular.

31. Factor the matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ so that $A = LL^T$, where L is lower triangular.
32. Determine directly the LL^T -factorization, in which L is a lower triangular matrix with positive diagonal elements, for the matrix

$$A = \begin{bmatrix} 4 & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{17}{16} & \frac{1}{4} \\ 1 & \frac{1}{4} & \frac{33}{64} \end{bmatrix}$$