Problem 1
(a) On most computers, the computation of $\sqrt{a}$ is based on Newton’s method. Set up the Newton iteration for solving $x^2 - a = 0$ and show that it can written in the form:

$$x_{k+1} = \frac{1}{2}(x_k + \frac{a}{x_k}).$$

(b) Show that

$$\sqrt{a} - x_{k+1} = \frac{1}{2x_k}(\sqrt{a} - x_k)^2.$$

Problem 2
Let $p$ and $q$ be real numbers with $0 < p < q$. Let

$$f(\xi_1, \xi_2) = (\xi_2 - p\xi_1^2)(\xi_2 - q\xi_1^2), \quad \text{and}$$

$$\dot{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

For any $d \in \mathbb{R}^n$, let $\theta(t) = f(\dot{x} + td)$.

(a) Show that for any non-zero $d$, $t = 0$ is a local minimizer of $\theta$.

(b) Show that

$$\dot{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is NOT a local minimizer of $f$.

Problem 3
For finding a root of $g(x) = x^5 - x - 1$, carry out (a) Newton’s method with the starting point $x_0 = 2$ (b) the secant method with starting points $x_0 = 2$ and $x_1 = 1$ until $|x_n - x_{n-1}| \leq 10^{-5}$.

Problem 4
Find all the descent directions of $f(x) = \max\{\xi_1 + \xi_2, 2\xi_1\xi_2\}$ at $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Problem 5
Let $f_1, f_2, \ldots, f_m$ be continuously differentiable functions on $\mathbb{R}^n$ and let

$$f(x) = \max\{f_1(x), \ldots, f_m(x)\}$$

Show that

$$f'(\dot{x}; d) = \max_{i \in I(\dot{x})} \nabla f_i(\dot{x})^T d$$

where $I(\dot{x}) = \{i | f_i(\dot{x}) = f(\dot{x})\}$. 