Problem 1
Let \( \{F_k\} \) be the Fibonacci sequence defined by \( F_0 = 1, \ F_1 = 1, \) and \( F_{k+1} = F_k + F_{k-1}. \) Show, by direct computation or any other proof, that
\[
F_k = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{k+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{k+1} \right].
\]

Problem 2
Find
\[
\lim_{n \to \infty} \frac{b_n - a_n}{b_{n+1} - a_{n+1}}
\]
for both the Golden-Section and the Fibonacci methods.

Problem 3
Let \( f : \mathbb{R} \to \mathbb{R} \) and \( f \in C^k, \) and let
\[
\frac{df(\hat{x})}{dx} = 0, \ldots, \frac{d^{k-1}f(\hat{x})}{dx^{k-1}} = 0, \quad \frac{d^k f(\hat{x})}{dx^k} > 0.
\]
Show that \( \hat{x} \) is a local minimizer if and only if \( k \) is even.

Problem 4
Use the Fibonacci Search to find the final interval for minimizing \( f(x) = 2x^2 - x + 1 \) over the interval \([0, 1]\) if only 7 function evaluations are allowed.

Problem 5
If the original interval has length 1 and we want to get an estimate \( \hat{z} \) to a minimizer \( \hat{x} \) within the range \( |z - \hat{x}| \leq 10^{-6}, \) how many function and derivative evaluations are needed for the following methods: (1) the bisection method; (2) The Golden Section Search and (3) the Fibonacci Search.