

Minimize  $8x_1 + 2x_2 + 3x_3$   
 s.t.  $-4x_1 - 2x_2 + y_1 = x_1 - 1$   
 $-2x_1 + 4x_2 + 4x_3 + y_2 = 3$   
 $3x_1 - 4x_2 - 2x_3 + y_3 = 4$   
 $x_i, y_i \geq 0$

Use dual simplex method:

0	8	2	3	0	0	0
-1	-4	-2	0	0	1	0
3	-2	4	4	0	1	0
4	3	-4	-2	0	0	1

choose pivot row  
 ratio test:  $\frac{C_j}{-B_{2j}}$ :  $-\frac{4}{-2} = 2, \frac{-2}{-1} = 2$

-1	4	0	3	1	0	0
1/2	2	1	0	0	0	0
1	-10	0	4	2	1	0
6	11	0	-2	-2	0	1

$X = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 1 \\ 6 \end{bmatrix}$  optimal solution (primal)

Dual solution:

$\hat{W} = B^{-T} C_B$ ;  $C_B = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ ;  $B^{-1} = \begin{bmatrix} -1/2 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

$\hat{W} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

$B^{-T} = \begin{bmatrix} -1/2 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\hat{W} = \begin{bmatrix} -1/2 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$  optimal solution (dual)

3. Min  $-x_1 + x_2$

s.t.  $2x_1 + x_2 \leq 2$   
 $x_1 - 2x_2 \leq 1$   
 $x_1 + x_2 \leq 4$   
 $x_1, x_2 \geq 0$

$\hat{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Is this optimal?

Assume it is optimal (and dual is feasible)  
 Use complementary slackness to check.

(1)  $(c - A^T \hat{w})_i \hat{x}_i = 0$      $A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 1 & 1 \end{bmatrix}$ ;  $A^T = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$

$i=1: \hat{x}_1 = 3 > 0 \Rightarrow (c - A^T \hat{w})_1 = 0$   
 $-1 + 2w_1 - w_2 - w_3 = 0$

$i=2: \hat{x}_2 = 1 > 0 \Rightarrow (c - A^T \hat{w})_2 = 0$   
 $1 - w_1 + 2w_2 - w_3 = 0$

(2) check constraints:  $\hat{w}_j (A \hat{x} - b)_j = 0$   
 (for  $j=1$ ):  $-2(3) + 1(1) = -5 < 2 \Rightarrow w_1 = 0$

$j=2: 1(3) - 2(1) = 1 = 1 \Rightarrow w_2 = ?$

$j=3: 1(3) + 1(1) = 4 = 4 \Rightarrow w_3 = ?$

Solve for  $\hat{w}$     (3) Now check  $c^T \hat{x} \stackrel{?}{=} b^T \hat{w}$

$2w_2 - w_3 = -1$   
 $-(-w_2 - w_3 = 1)$   
 $3w_2 = -2$

$w_2 = -\frac{2}{3}$

$w_3 = 2(-\frac{2}{3}) + 1 = -\frac{1}{3}$

$c^T \hat{x} = -3 + 1 = -2$   
 $b^T \hat{w} = 2(0) + 1(-\frac{2}{3}) + 4(-\frac{1}{3}) = -\frac{6}{3} = -2$

$\hat{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$  is optimal (primal)  
 $\hat{w} = \begin{bmatrix} 0 \\ -\frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}$  is optimal dual

4. Min  $\alpha x_1 + 13x_2 - \alpha x_3 - 3x_4$  (5.5, 4) H  
 s.t.  $x_1 + x_2 + 2x_3 + 3x_4 = \beta$  (standard form)  
 $2x_1 + x_2 + 3x_3 + 4x_4 = \beta$   
 $x_i \geq 0$

Basic is primal infeasible but dual feasible.  
 Want  $x_3, x_4$  in basis.

$B = \begin{bmatrix} -2 & 3 \\ 3 & 4 \end{bmatrix}; B^{-1} = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$

(1) all zero<sup>th</sup> row  $\geq 0$  (column  $\geq 0$ )

$$B^{-1}b = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ \beta \end{bmatrix} = \begin{bmatrix} -4 - 3\beta \\ -3 - 2\beta \end{bmatrix}$$

$$-4 - 3\beta \geq 0 \Rightarrow -3\beta \geq 4 \quad -3 - 2\beta \geq 0$$

$$\beta \leq -\frac{4}{3} \quad \beta \leq -\frac{3}{2}$$

(2) all reduced costs  $\geq 0$  ( $c_i - c_B^T B^{-1} a_i \geq 0$ )

$$c_B^T B^{-1} = [-\alpha \quad -3] \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix} = [-4\alpha - 9 \quad 3\alpha + 6]$$

$i=1: \alpha - [-4\alpha - 9, 3\alpha + 6] \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \alpha - (-4\alpha - 9 + 6\alpha + 12)$

$$= \alpha(1 + 4 - 6) - 3 = -\alpha - 3 \geq 0$$

$$\alpha \leq -3$$

$i=2: 13 - [-4\alpha - 9, 3\alpha + 6] \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 13 - (-4\alpha - 9 - 3\alpha - 6)$

$$= 7\alpha + 28 \geq 0$$

$$\alpha \geq -4$$

$$\boxed{-4 \leq \alpha \leq -3}$$