1. Min $8x_1 + 2x_2 + 3x_3$

s.t. $-4x_1 - 2x_2 + y_1 = -1$

$-2x_1 + 4x_2 + 4x_3 + y_2 = 3$

$3x_1 - 4x_2 - 2x_3 + y_3 = 4$

$x_i, y_i \geq 0$

Use dual simplex method:

0	8	2	3	0	0	0
-1	-4	-2	0	1	0	0
3	-2	4	4	0	1	0
4	3	-4	-2	0	0	1

← choose pivot row

ratio test: $\frac{c_i}{-B_i}$: $\frac{-4}{1}$, $\frac{-2}{1}$

-1	4	0	3	1	0	0
1/2	2	1	0	-1/2	0	0
1	-10	0	4	2	1	0
6	11	0	-2	-2	0	1

$$x = \begin{bmatrix} 0 \\ 1/2 \\ 0 \\ 1 \\ 6 \end{bmatrix}$$ optimal solution (primal)

Dual solution:

$$\hat{w} = B^{-T} c_B \; ; \quad c_B = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \; ; \quad B^{-1} = \begin{bmatrix} -1/2 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$B^{-T} = \begin{bmatrix} -1/2 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\hat{w} = \begin{bmatrix} -1/2 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$ optimal solution (dual)

3. Min $-x_1 + x_2$

s.t. $-2x_1 + x_2 \le 2$

$x_1 - 2x_2 \le 1$

$x_1 + x_2 \le 4$

$x_1, x_2 \ge 0$

$\hat{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ Is this optimal?

Assume it is optimal (and dual is feasible)
Use complementary slackness to check.

(1) $(c - A^T\hat{w})_i \hat{x}_i = 0$ $\quad A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \\ 1 & 1 \end{bmatrix}; \; A^T = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}$

$i=1: \hat{x}_1 = 3 > 0 \Rightarrow (c - A^T\hat{w})_1 = 0$

$-1 + 2w_1 - w_2 - w_3 = 0$

$i=2: \hat{x}_2 = 1 > 0 \Rightarrow (c - A^T\hat{w})_2 = 0$

$1 - w_1 + 2w_2 - w_3 = 0$

(2) check constraints: $\hat{w}_j (A\hat{x} - b)_j = 0$

$j=1: -2(3) + 1(1) = -5 < 2 \Rightarrow w_1 = 0$

$j=2: 1(3) - 2(1) = 1 = 1 \Rightarrow w_2 = ?$

$j=3: 1(3) + 1(1) = 4 = 4 \Rightarrow w_3 = ?$

Solve for $\hat{w}$:

$2w_2 - w_3 = -1$

$-(-w_2 - w_3 = 1)$

$3w_2 = -2$

$w_2 = \frac{-2}{3}$

$w_3 = 2\left(\frac{-2}{3}\right) + 1 = \frac{-1}{3}$

(3) Now check $c^T\hat{x} \overset{?}{=} b^T\hat{w}$

$c^T\hat{x} = -3 + 1 = -2$

$b^T\hat{w} = 2(0) + 1\left(\frac{-2}{3}\right) + 4\left(\frac{-1}{3}\right) = \frac{-6}{3} =$

$\hat{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is optimal (primal)

$\hat{w} = \begin{bmatrix} 0 \\ -2/3 \\ -1/3 \end{bmatrix}$ is optimal dual

4. Min $\alpha x_1 + 13x_2 - \alpha x_3 - 3x_4$

s.t. $x_1 + x_2 - 2x_3 + 3x_4 = -1$

$2x_1 - x_2 - 3x_3 + 4x_4 = \beta$

$x_i \geq 0$

Want x_3, x_4 in basis.

0	α	13	$-\alpha$	-3
-1	1	1	-2	3
β	2	-1	-3	4

$B = \begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$; $B^{-1} = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}$

(1) all zeroth row ≥ 0

$B^{-1}b = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ \beta \end{bmatrix} = \begin{bmatrix} -4 - 3\beta \\ -3 - 2\beta \end{bmatrix}$

$-4 - 3\beta \geq 0 \Rightarrow -3\beta \geq 4$ $\qquad$ $-3 - 2\beta \geq 0$

$\beta \leq -\frac{4}{3}$ $\qquad$ $\boxed{\beta \leq \frac{-3}{2}}$

(2) all reduced costs ≥ 0 $\quad (c_i - c_B^T B^{-1} a_i \geq 0)$

$c_B^T B^{-1} = [-\alpha \; -3] \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix} = [-4\alpha - 9 \quad 3\alpha + 6]$

$i=1$: $\alpha - [-4\alpha - 9 \quad 3\alpha + 6] \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \alpha - (-4\alpha - 9 + 6\alpha + 12)$

$= \alpha(1 + 4 - 6) - 3 = -\alpha - 3 \geq$

$\alpha \leq -$

$i=2$: $13 - [-4\alpha - 9, 3\alpha + 6] \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 13 - (-4\alpha - 9 - 3\alpha - 6)$

$= 7\alpha + 28 \geq 0$

$\alpha \geq -4$

$\boxed{-4 \leq \alpha \leq -3}$