Special thanks to Aubin for providing the excellent answer to certain problems

Prob. 1

First we note that this is in standard form. We do not need to introduce any slack variables. Next we note that the constraints are not linearly independent.

$$\begin{aligned} &2(-x_1 - x_2 - 2x_3 + 3x_4 = -2) \\ &+(2x_1 + x_2 + 3x_3 - 2x_4 = 3) \\ &= (-x_2 - x_3 + 4x_4 = -1) \end{aligned}$$

Therefore, we dismiss the extraneous third constraint. At the same time, we multiply the first constraint by -1. Our problem then becomes

$$\begin{aligned} \min \quad & -3x_1 - x_2 \\ \text{subject to} \quad & x_1 + x_2 + 2x_3 - 3x_4 = 2 \\ & 2x_1 + x_2 + 3x_3 - 2x_4 = 3 \\ & x_i \geq 0, \quad i = 1,2,3,4. \end{aligned}$$

In the Phase 1 calculation, we introduce to artificial variables y_1, y_2, in our attempt to find a vertex of the original function. Here we want to solve

$$\begin{aligned} \min \quad & y_1 + y_2 \\ \text{subject to} \quad & x_1 + x_2 + 2x_3 - 3x_4 + y_1 = 2 \\ & 2x_1 + x_2 + 3x_3 - 2x_4 + y_2 = 3 \\ & x_i \geq 0, \quad i = 1,2,3,4 \\ & y_i \geq 0, \quad i = 1,2 \end{aligned}$$

We solve this with the method of tableaux.

$$T_0 = \begin{array}{|c|cccccc|} \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 2 & 1 & 1 & 2 & -3 & 1 & 0 \\ 3 & 2 & 1 & 3 & -2 & 0 & 1 \\ \hline \end{array}$$

This has the original costs in the zeroeth row. So we first adjust this to find the reduced costs.

$$T_1 = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} T_0 = \begin{array}{|c|cccccc|} \hline -5 & -3 & -2 & -5 & 5 & 0 & 0 \\ \hline 2 & 1 & 1 & 2 & -3 & 1 & 0 \\ 3 & 2 & 1 & 3 & -2 & 0 & 1 \\ \hline \end{array}$$

We have several choices for the column on which we pivot. Choosing at random we pivot on x_2. By the ratio test, we must pivot on the row corresponding to y_1.

$$T_2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} T_1 = \begin{array}{|c|cccccc|} \hline -1 & -1 & 0 & -1 & -1 & 2 & 0 \\ \hline 2 & 1 & 1 & 2 & -3 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & -1 & 1 \\ \hline \end{array}$$

Next we choose to pivot on x_4. Because there is only one positive entry in that column, we pivot on the bottom row.

$$T_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} T_2 = \begin{array}{|c|cccccc|} \hline 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ \hline 5 & 4 & 1 & 5 & 0 & -2 & 3 \\ 1 & 1 & 0 & 1 & 1 & -1 & 1 \\ \hline \end{array}$$

We are done with the Phase 1 calculation. We have found a vertex of the original problem. We return to the original problem by dropping the last two columns and reintroducing the original costs.

$$T_0 = \begin{array}{|c|cccc|} \hline 0 & -3 & -1 & 0 & 0 \\ \hline 5 & 4 & 1 & 5 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ \hline \end{array}$$

Again, the first step is to find the reduced costs.

$$T_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} T_0 = \begin{array}{|c|cccc|} \hline 5 & 1 & 0 & 5 & 0 \\ \hline 5 & 4 & 1 & 5 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ \hline \end{array}$$

We see that at this vertex, all the reduced costs of the nonbasic variables are positive. This means that this is the optimal solution. Our solution is

$$x_1 = 0,\ x_2 = 5,\ x_3 = 0,\ x_4 = 1 \qquad \text{with optimal cost} \qquad -3x_1 - x_2 = -5.$$

Prob. 2

We introduce three slack variables u_1, u_2, u_3 to put this into standard form.

$$\begin{array}{rl} \min & -3x_1 + x_2 \\ \text{subject to} & 2x_1 + x_2 - u_1 = 2 \\ & x_1 + 3x_2 + u_2 = 3 \\ & x_2 + u_3 = 4 \\ & x_i \geq 0, \quad i = 1, 2 \\ & u_i \geq 0, \quad i = 1, 2, 3. \end{array}$$

The corresponding tableau is

$$T_0 = \begin{array}{|c|rrrrr|} \hline 0 & -3 & 1 & 0 & 0 & 0 \\ \hline 2 & 2 & 1 & -1 & 0 & 0 \\ 3 & 1 & 3 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 0 & 1 \\ \hline \end{array}$$

This is not a simplex tableau. We are missing the first standard basis vector. We introduce one artificial variable and solve a Phase 1 problem to find a vertex.

$$S_0 = \begin{array}{|c|rrrrrr|} \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 2 & 2 & 1 & -1 & 0 & 0 & 1 \\ 3 & 1 & 3 & 0 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline \end{array}$$

We calculate the reduced costs

$$S_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} S_0 = \begin{array}{|c|rrrrrr|} \hline -2 & -2 & -1 & 1 & 0 & 0 & 0 \\ \hline 2 & 2 & 1 & -1 & 0 & 0 & 1 \\ 3 & 1 & 3 & 0 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline \end{array}$$

We choose to pivot on x_1. The ratio test says that we need to pivot on the 2 in that column.

$$S_2 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & -0.5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} S_1 = \begin{array}{|c|rrrrrr|} \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 0.5 & -0.5 & 0 & 0 & 0.5 \\ 2 & 0 & 2.5 & 0.5 & 1 & 0 & -0.5 \\ 4 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline \end{array}$$

We can remove the artificial variable now and reintroduce the original objective function.

$$T_1 = \begin{array}{|c|rrrrr|} \hline 0 & -3 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 0.5 & -0.5 & 0 & 0 \\ 2 & 0 & 2.5 & 0.5 & 1 & 0 \\ 4 & 0 & 1 & 0 & 0 & 1 \\ \hline \end{array}$$

We need to calculate the reduced costs.

$$T_2 = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} T_1 = \begin{array}{|c|rrrrr|} \hline 3 & 0 & 2.5 & -1.5 & 0 & 0 \\ \hline 1 & 1 & 0.5 & -0.5 & 0 & 0 \\ 2 & 0 & 2.5 & 0.5 & 1 & 0 \\ 4 & 0 & 1 & 0 & 0 & 1 \\ \hline \end{array}$$

There is only one choice of pivot. We must pivot on x_3 in the second row.

$$T_3 = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} T_2 = \begin{array}{|c|rrrrr|} \hline 9 & 0 & 10 & 0 & 3 & 0 \\ \hline 3 & 1 & 3 & 0 & 1 & 0 \\ 4 & 0 & 5 & 1 & 2 & 0 \\ 4 & 0 & 1 & 0 & 0 & 1 \\ \hline \end{array}$$

This tableau corresponds to an optimal solution.

$$x_1 = 3,\ x_2 = 0 \qquad \text{with optimal value} \qquad -3x_1 + x_2 = -9.$$

Prob. 3

Let M be a very large positive value. Then we solve the LP by the following tableau manipulations.

$$T_0 = \begin{array}{|c|cccccc|}\hline 0 & -3 & 1 & 0 & 0 & 0 & M \\ \hline 2 & 2 & 1 & -1 & 0 & 0 & 1 \\ 3 & 1 & 3 & 0 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline \end{array}$$

$$T_1 = \begin{bmatrix} 1 & -M & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} T_0 = \begin{array}{|c|ccccccc|}\hline -2M & -3-2M & 1-M & M & 0 & 0 & 0 \\ \hline 2 & 2 & 1 & -1 & 0 & 0 & 1 \\ 3 & 1 & 3 & 0 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline \end{array}$$

$$T_2 = \begin{bmatrix} 1 & 0.5(3+2M) & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & -0.5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} T_1 = \begin{array}{|c|cccccc|}\hline 3 & 0 & 5/2 & -3/2 & 0 & 0 & 3/2+M \\ \hline 1 & 1 & 1/2 & -1/2 & 0 & 0 & 1/2 \\ 2 & 0 & 5/2 & 1/2 & 1 & 0 & -1/2 \\ 4 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline \end{array}$$

$$T_3 = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} T_2 = \begin{array}{|c|cccccc|}\hline 9 & 0 & 10 & 0 & 3 & 0 & M \\ \hline 3 & 1 & 3 & 0 & 1 & 0 & 0 \\ 4 & 0 & 5 & 1 & 2 & 0 & -1 \\ 4 & 0 & 1 & 0 & 0 & 1 & 0 \\ \hline \end{array}$$

We end up with the same solution as above.

$$x_1 = 3,\ x_2 = 0 \qquad \text{with optimal value} \qquad -3x_1 + x_2 = -9.$$

Prob. 4

Suppose that $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. Let $e_M = (M, M, \ldots, M)^\top \in \mathbb{R}^m$. Then if the original problem is

$$\begin{aligned} \min \quad & c^\top x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

then the corresponding big-M formulation is

$$\begin{aligned} \min \quad & c^\top x + e_M^\top y \\ \text{subject to} \quad & Ax + Iy = b \\ & x,\ y \geq 0 \end{aligned}$$

where I is the $m \times m$ identity matrix.

Suppose that, as stated in the problem, (x, y) with $y = 0$ is found as an optimal solution for the big-M problem. We will write $\mathbf{0} \in \mathbb{R}^m$ for the vector of all zeros. Let $\bar{x}$ be any other feasible point for the original problem. Then

$$b = A\bar{x} = A\bar{x} + 0 = A\bar{x} + I\mathbf{0}$$

That is, $(\bar{x}, \mathbf{0})$ is a feasible point for the big-M problem. Because $(x, \mathbf{0})$ is optimal for the big-M problem,

$$c^\top x = c^\top x + 0 = c^\top x + e_M^\top \mathbf{0} \leq c^\top \bar{x} + e_M^\top \mathbf{0} = c^\top \bar{x} + 0 = c^\top \bar{x}$$

Thus, x is optimal for the original problem.