Special thanks to Aubin for providing the excellent answer to certain problems

Problem 2

We use the method of tableaux to compute these vertices. Note that the problem is already in standard form. Then the first tableau is

$$T_0 = \left[\begin{array}{c|cccc} 8 & 4 & -1 & 1 & 0 \\ 6 & -2 & 3 & 0 & 1 \end{array}\right]$$

Because 8 and 6 are both nonnegative, this corresponds to a vertex.

$$v_0 = \begin{bmatrix} 0 \\ 0 \\ 8 \\ 6 \end{bmatrix}$$

We can also read off the basic directions from the tableau.

$$d_0^1 = \begin{bmatrix} 1 \\ 0 \\ -4 \\ 2 \end{bmatrix} \qquad d_0^2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -3 \end{bmatrix}$$

Then the reduced costs are

$$c^\top d_0^1 = (1,\ -1,\ 3,\ -5) \cdot (1,\ 0,\ -4,\ 2) = 1 - 12 - 10 = -21$$
$$c^\top d_0^2 = (1,\ -1,\ 3,\ -5) \cdot (0,\ 1,\ 1,\ -3) = -1 + 3 + 15 = 17.$$

Since one of these are negative, this is not an optimal vertex. To move to an adjacent vertex, we can pivot on either the first or second column. We will do both of these and label the new tablueax T_{-1} and T_1 respectively. In each case there is only one possible pivot because there is only one positve value in each of those columns.

$$T_{-1} = \left[\begin{array}{c|cccc} 2 & 1 & -1/4 & 1/4 & 0 \\ 10 & 0 & 5/2 & 1/2 & 1 \end{array}\right] \qquad T_1 = \left[\begin{array}{c|cccc} 10 & 10/3 & 0 & 1 & 1/3 \\ 2 & -2/3 & 1 & 0 & 1/3 \end{array}\right]$$

The corresponding vertices are

$$v_{-1} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 10 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 0 \\ 2 \\ 10 \\ 0 \end{bmatrix}$$

The corresponding basic directions are

$$d_{-1}^1 = \begin{bmatrix} 1/4 \\ 1 \\ 0 \\ -5/2 \end{bmatrix} \quad d_{-1}^2 = \begin{bmatrix} -1/4 \\ 0 \\ 1 \\ -1/2 \end{bmatrix} \qquad d_1^1 = \begin{bmatrix} 1 \\ 2/3 \\ -10/3 \\ 0 \end{bmatrix} \quad d_1^2 = \begin{bmatrix} 0 \\ -1/3 \\ -1/3 \\ 1 \end{bmatrix}$$

The costs at v_{-1} are

$$c^\top d_{-1}^1 = (1,\ -1,\ 3,\ -5) \cdot (1/4,\ 1,\ 0,\ -5/2) = 47/4$$
$$c^\top d_{-1}^2 = (1,\ -1,\ 3,\ -5) \cdot (-1/4,\ 0,\ 1,\ -1/2) = 21/4.$$

Since both of these are positive, we know that v_{-1} is optimal. By the first exercise above, we know that v_{-1} is the unique optimum because these reduced costs are strictly positive.

The costs at v_1 are

$$c^\top d_1^1 = (1,\ -1,\ 3,\ -5) \cdot (1, 2/3, -10/3, 0) = -29/3$$
$$c^\top d_1^2 = (1,\ -1,\ 3,\ -5) \cdot (0, -1/3, -1/3, 1) = -17/3$$

Since these costs are negative, we conclude that v_1 it not optimal. Of course, we have already found the unique optimum, so we know that no further vertex will be optimal.

Next we want to pivot from T_{-1} to get a new vertex. If we pivot on x_3, we will return to T_0. So we must pivot on x_2. Since there is only one positive number in that column, we have only one choice of pivot. The new tableau we will call T_{-2}.

$$T_{-2} = \left[\begin{array}{c|cccc} 3 & 1 & 0 & 3/10 & 1/10 \\ 4 & 0 & 1 & 1/5 & 2/5 \end{array}\right]$$