Special thanks to Aubin for providing the excellent answer to certain problems

Problem 1

First we transform the polyhedron into standard form by introducing two slack variables.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 12 \\ x_1 - x_2 + x_4 &= 3 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \\ x_3 &\geq 0 \\ x_4 &\geq 0 \end{aligned}$$

We now convert this to a tableau. Then we cycle through the vertices by choosing pivots that : and satisfy the ratio test. We will write $\bar{x} = (x_1, x_2, x_3, x_4)^\top$, which corresponds to the point x in the original linear program.

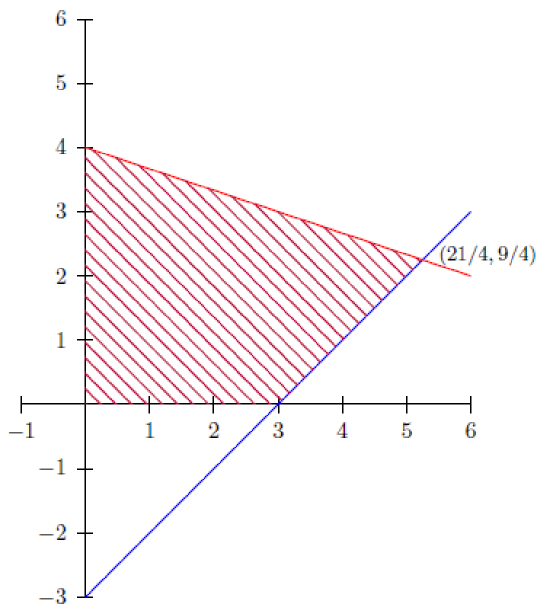
$$T_0 = \left[\begin{array}{c|cccc} 12 & 1 & 3 & 1 & 0 \\ 3 & 1 & -1 & 0 & 1 \end{array}\right] \Longrightarrow \bar{x} = \begin{bmatrix} 0 \\ 0 \\ 12 \\ 3 \end{bmatrix} \Longrightarrow x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T_1 = \left[\begin{array}{c|cccc} 4 & 1/3 & 1 & 1/3 & 0 \\ 7 & 4/3 & 0 & 1/3 & 1 \end{array}\right] \Longrightarrow \bar{x} = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 7 \end{bmatrix} \Longrightarrow x = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$T_2 = \left[\begin{array}{c|cccc} 9/4 & 0 & 1 & 1/4 & -1/4 \\ 21/4 & 1 & 0 & 1/4 & 3/4 \end{array}\right] \Longrightarrow \bar{x} = \begin{bmatrix} 21/4 \\ 9/4 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow x = \begin{bmatrix} 21/4 \\ 9/4 \end{bmatrix}$$

$$T_3 = \left[\begin{array}{c|cccc} 9 & 0 & 4 & 1 & -1 \\ 3 & 1 & -1 & 0 & 1 \end{array}\right] \Longrightarrow \bar{x} = \begin{bmatrix} 3 \\ 0 \\ 9 \\ 0 \end{bmatrix} \Longrightarrow x = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

When we pivot from this tableau, we get back to the beginning.



The feasible region is the hashed polyhedron in the figure above. We see that with our choice of pivots above, we travelled around the polyhedron starting at the origin and continuing in a clockwise direction.

It remains to find the optimal points among the vertices. Recall that we are hoping to maximize the objective function $f(x) = f(x_1, x_2) = x_1 + 2x_2$.

x	$f(x)$
$(0, 0)$	0
$(0, 4)$	8
$(21/4, 9/4)$	$39/4 = 9.75$
$(3, 0)$	3

So the optimal value is 9.75 found at $(21/4, 9/4)$.

Problem 2

Let

$$F = F(\hat{x}, P) = \{d \in \mathbb{R}^n : \exists\varepsilon > 0 \text{ such that } \hat{x} + \lambda d \in P,\ \forall\lambda \in [0, \varepsilon]\}$$

be the set of feasible directions for P at $\hat{x}$. Also let

$$S = \{d : Ad = 0.\ Dd \leq 0\}.$$

We want to show that $S = F$.

First let $d \in F$. Choose the corresponding $\varepsilon > 0$. Then we know that $\hat{x} + \varepsilon d \in P$. Then we know

$$b = A(\hat{x} + \varepsilon d) = A\hat{x} + \varepsilon A d = b + \varepsilon A d \quad \Longrightarrow \quad 0 = \varepsilon A d \quad \Longrightarrow \quad 0 = Ad$$

since $\varepsilon > 0$. Also we have

$$f \geq D(\hat{x} + \varepsilon d) = D\hat{x} + \varepsilon D d = f + \varepsilon D d \quad \Longrightarrow \quad 0 \geq \varepsilon D d \quad \Longrightarrow \quad 0 \geq Dd$$

since $\varepsilon > 0$. Thus $d \in S$, and we have shown that $F \subseteq S$.

Next suppose that $d \in S$. We want to find a corresponding $\varepsilon > 0$ to show that $d \in F$. Note

$$A(\hat{x} + \varepsilon d) = A\hat{x} + \varepsilon A d = b + \varepsilon A d = b$$

regardless of the value of ε since $Ad = 0$. Next note that

$$D(\hat{x} + \varepsilon d) = D\hat{x} + \varepsilon D d = f + \varepsilon D d \leq f$$

since $Dd \leq 0$ and for any $\varepsilon \geq 0$.

Now we consider the last constraint, $Ex \leq g$ for $x \in P$. This will restrict the size of ε. Recall that $E\hat{x} < g$. We want to choose ε so that

$$g \geq E(\hat{x} + \varepsilon d) = E\hat{x} + \varepsilon E d \quad \Longleftrightarrow \quad g - E\hat{x} \geq \varepsilon E d$$

Since $E\hat{x} < g$, $g - E\hat{x} > 0$. That is, every component of $g - E\hat{x}$ is strictly positive. If every component of Ed is nonpositive, then $\varepsilon > 0$ can be as large as possible and the inequality will always hold. In that case, we take $\varepsilon = 1$ just to have some concrete value. Otherwise, suppose that at least one component of Ed is positive. Take

$$\varepsilon = \min\left\{\frac{(g - E\hat{x})_i}{(Ed)_i} : (Ed)_i > 0\right\}$$

Since at least one component of Ed is positive, and every component of $g - E\hat{x}$ is positive, we know that $\varepsilon > 0$. With this definition, we have for every index i such that $(Ed)_i > 0$

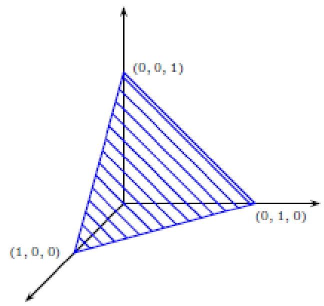
$$(g - E\hat{x})_i \geq \varepsilon (Ed)_i \geq \lambda (Ed)_i \quad \forall \lambda \in [0, \varepsilon].$$

And again, for the components with $(Ed)_i \leq 0$, $(g - E\hat{x})_i \geq \lambda (Ed)_i$ $\forall \lambda > 0$. So for this definition of ε, $\hat{x} + \lambda d \in P$ for all $\lambda \in [0, \varepsilon]$.

This shows that $d \in F$, so that $S \subseteq F$.

Thus $S = F$.

Problem 3



The set P is the section of the hyperplane shown above. The three vertices are $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Let

$$d^1 = (1,0,0) - (0,0,1) = (1,0,-1) \quad \text{and} \quad d^2 = (0,1,0) - (0,0,1) = (0,1,-1).$$

These are clearly both feasible directions for P at $(0,0,1)$, because $d = y - x$ is a feasible direction for P at x for any $y \in P \setminus \{x\}$ since P is convex.

More than that, these two directions generate $F = F(P,(0,0,1))$, the cone of feasible directions of P at $(0,0,1)$. We know from lecture that

$$F = \{d = (d_1, d_2, d_3) : d_1 + d_2 + d_3 = 0,\ d_1 \geq 0,\ d_2 \geq 0\}$$

Note that this is a two-dimensional set because it involves the solution of one linear function in $\mathbb{R}^3$. That is, it is a portion of a hyperplane in $\mathbb{R}^3$, so it has dimension $3 - 1 = 2$. By construction, d^1, $d^2 \in F$. Moreover, d^1 and d^2 are linearly independent so that their span is two-dimensional. Since F has dimension two, these two vectors generate the entire set.

We could also find the generators by observing the corresponding tableau. This problem is already in standard form. Since there is only one constraint, the tableau has but one row.

$$T = \left[\begin{array}{c|ccc} 1 & 1 & 1 & 1 \end{array}\right]$$

In this case, we take the last "column" to be the basic variable. Then there are two non-basic variables, and we know that the two generators for the cone of feasible directions are

$$d^1 = \begin{pmatrix} e_1 \\ -B^{-1}A_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad d^2 = \begin{pmatrix} e_2 \\ -B^{-1}A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

which is what we have above.

Knowing the generators, we can write the cone of feasible directions in this slightly different way,

$$\begin{aligned} F &= \{d = (d_1, d_2, d_3) : d_1 + d_2 + d_3 = 0,\ d_1 \geq 0,\ d_2 \geq 0\} \\ &= \{d = (d_1, d_2, -d_1 - d_2) : d_1 \geq 0,\ d_2 \geq 0\} \end{aligned}$$