Special thanks to Aubin for providing the excellent answer to certain problems

Problem 1

First, we number the constraints for convenience.

1	$3x_1 + x_2 + x_3 \leq 5$
2	$x_1 + x_2 - x_3 = 1$
3	$x_2 \geq -3$
4	$x_1 \geq 0$
5	$x_3 \geq 0$

A vertex must satisfy 2, the only equality constraint. Since all of these contraints are linearly independent, we need to choose two other contraints. There are $\binom{4}{2} = 6$ ways to do this. The solutions will be a basic solution. We need to check the remaining constraints to see if the basic solution is also feasible. In that case, the basic feasible solution is a vertex. We go through these options one-by-one.

Active Contraints: 2, 4, 5

$$\left.\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ x_1 &= 0 \\ x_3 &= 0 \end{aligned}\right\} \implies \left.\begin{aligned} x_2 &= 1 \\ x_1 &= 0 \\ x_3 &= 0 \end{aligned}\right\} \implies x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

This basic solution also meets the other contraints because

$$\begin{aligned} 3x_1 + x_2 + x_3 = 1 &\leq 5 \\ x_2 = 1 &\geq -3. \end{aligned}$$

So this basic solution is a vertex.

Active Contraints: 2, 4, 1

$$\left.\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ x_1 &= 0 \\ 3x_1 + x_2 + x_3 &= 5 \end{aligned}\right\} \implies \left.\begin{aligned} x_2 - x_3 &= 1 \\ x_1 &= 0 \\ x_2 + x_3 &= 5 \end{aligned}\right\} \implies x = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$$

This basic solution also meets the other contraints because

$$\begin{aligned} x_2 &= 3 \geq -3 \\ x_3 &= 2 \geq 0. \end{aligned}$$

So this basic solution is a vertex.

Active Contraints: 2, 4, 3

$$\left.\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ x_1 &= 0 \\ x_2 &= -3 \end{aligned}\right\} \implies \left.\begin{aligned} -3 - x_3 &= 1 \\ x_1 &= 0 \\ x_2 &= -3 \end{aligned}\right\} \implies x = \begin{bmatrix} 0 \\ -3 \\ -4 \end{bmatrix}$$

This basic solution does not meet contraint 5 because $-3 < 0$. So this basic solution is not a vertex.

Active Contraints: 2, 5, 1

$$\left.\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ x_3 &= 0 \\ 3x_1 + x_2 + x_3 &= 5 \end{aligned}\right\} \implies \left.\begin{aligned} x_1 + x_2 &= 1 \\ x_3 &= 0 \\ 3x_1 + x_2 &= 5 \end{aligned}\right\} \implies x = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

This basic solution also meets the other contraints because

$$\begin{aligned} x_2 &= -1 \geq -3 \\ x_1 &= 2 \geq 0. \end{aligned}$$

So this basic solution is a vertex.

Active Contraints: 2, 5, 3

$$\left.\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ x_3 &= 0 \\ x_2 &= -3 \end{aligned}\right\} \implies \left.\begin{aligned} x_1 - 3 &= 1 \\ x_3 &= 0 \\ x_2 &= -3 \end{aligned}\right\} \implies x = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$$

This basic solution does not meet contraint 1 because

$$3x_1 + x_2 + x_3 = 12 - 3 = 9 > 5.$$

So this basic solution is not a vertex.

Active Contraints: 2, 1, 3

$$\left.\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ 3x_1 + x_2 + x_3 &= 5 \\ x_2 &= -3 \end{aligned}\right\} \implies \left.\begin{aligned} x_1 - 3 - x_3 &= 1 \\ 3x_1 - 3 + x_3 &= 5 \\ x_2 &= -3 \end{aligned}\right\} \implies x = \begin{bmatrix} 3 \\ -3 \\ -1 \end{bmatrix}$$

This basic solution does not meet contraint 5 because $x_3 = -1 < 0$. So this basic solution is not a vertex.

Now we have all the vertices. We know that two vertices are adjacent if they share $n-1 = 2$ active contraints. Below we display which vertices are adjacent.

Problem 2

First we convert the problem to standard form.

$$S = \left\{ x \in \mathbb{R}^3 \mid \begin{array}{r} 3x_1 + \bar{x}_2 + x_3 + s = 8 \\ x_1 + \bar{x}_2 - x_3 = 4 \\ \bar{x}_2 \geq 0 \\ x_1 \geq 0 \\ x_3 \geq 0 \\ s \geq 0 \end{array} \right\}$$

where s is a slack variable and $\bar{x}_2 = x_2 + 3$. I will write $\bar{x}$ for the four-tuple $(x_1, \bar{x}_2, x_3, s)^\top$ and x for the three-tuple $(x_1, x_2, x_3)^\top$. By manipulating a tableau, we will find the same solutions that we had in the above problem.

$$\left[\begin{array}{c|cccc} 8 & 3 & 1 & 1 & 1 \\ 4 & 1 & 1 & -1 & 0 \end{array}\right] \quad \text{(The initial tableau)}$$

$$\left[\begin{array}{c|cccc} 12 & 4 & 2 & 0 & 1 \\ -4 & -1 & -1 & 1 & 0 \end{array}\right] \Longrightarrow \bar{x} = \begin{bmatrix} 0 \\ 0 \\ -4 \\ 12 \end{bmatrix} \Longrightarrow x = \begin{bmatrix} 0 \\ -3 \\ -4 \end{bmatrix} \quad \text{Not feasible}$$

$$\left[\begin{array}{c|cccc} 4 & 2 & 0 & 2 & 1 \\ 4 & 1 & 1 & -1 & 0 \end{array}\right] \Longrightarrow \bar{x} = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 4 \end{bmatrix} \Longrightarrow x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{Fesible. A vertex.}$$

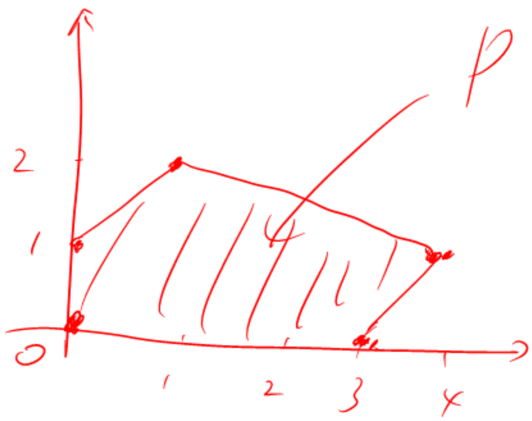
$$\left[\begin{array}{c|cccc} -4 & 0 & -2 & 4 & 1 \\ 4 & 1 & 1 & -1 & 0 \end{array}\right] \Longrightarrow \bar{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ -4 \end{bmatrix} \Longrightarrow x = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} \quad \text{Not fesible.}$$

$$\left[\begin{array}{c|cccc} -1 & 0 & -1/2 & 1 & 1/4 \\ 3 & 1 & 1/2 & 0 & 1/4 \end{array}\right] \Longrightarrow \bar{x} = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix} \Longrightarrow x = \begin{bmatrix} 3 \\ -3 \\ -1 \end{bmatrix} \quad \text{Not fesible.}$$

$$\left[\begin{array}{c|cccc} 2 & 1 & 0 & 1 & 1/2 \\ 6 & 2 & 1 & 0 & 1/2 \end{array}\right] \Longrightarrow \bar{x} = \begin{bmatrix} 0 \\ 6 \\ 2 \\ 0 \end{bmatrix} \Longrightarrow x = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} \quad \text{Fesible. A vertex.}$$

$$\left[\begin{array}{c|cccc} 2 & 1 & 0 & 1 & 1/2 \\ 2 & 0 & 1 & -2 & -1/2 \end{array}\right] \Longrightarrow \bar{x} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow x = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \quad \text{Fesible. A vertex.}$$

Problem 3



P is the shaded area

vertices of P: $\begin{pmatrix}0\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\end{pmatrix}$ $\begin{pmatrix}2\\2\end{pmatrix}$, $\begin{pmatrix}4\\1\end{pmatrix}$, $\begin{pmatrix}3\\0\end{pmatrix}$

$$P = \{ x \mid x_1 - 2x_2 \geq -2,\ x_1 + 2x_2 \leq 6,\ x_1 - x_2 \leq 3,\ x_1 \geq 0\ \ x_2 \geq 0 \}$$

Let $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ -1 & 2 \\ 1 & 2 \\ 1 & -1 \end{pmatrix}$ $b = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 6 \\ 3 \end{pmatrix}$

then $P = \{ x : Ax \leq b \}$

Problem 4

$$\max \sum_{i=1}^{n} e_i^T (u_i - v_i)$$

s.t. $A u_i \leq b$ $i = 1, 2, \cdots n$

$A v_i \leq b$ $i = 1, 2, \cdots n$

$e_i = (0, 0, \cdots 0, 1\ 0, 0 \cdots 0)^T$