Special thanks to Aubin for providing the excellent answer.

Exercise 1 *Formulate a mathematical model for the following problem. A food store packages three types of snack foods– Chewy, Crunchy, and Nutty – by mixing sunflower seeds, raisins, and peanuts. The specifications of each mixture are given in the following table.*

Mixture	*Sunflower Seeds*	*Raisins*	*Peanuts*	*Price per kg.*
Chewy	*none*	*at least 60%*	*at most 20%*	*$2.00*
Crunchy	*at least 60%*	*none*	*at least 10%*	*$1.60*
Nutty	*at most 20%*	*none*	*at least 60%*	*$1.20*

The suppliers of the ingredients can deliver each week at most 100 kgs of sunflower seeds at $1.00 per kg, 80 kgs of raisins at $1.50 per kg, and 60 kgs of peanuts at $0.80 per kg. Determine a mixture scheme that will maximize the store's profits

First I will enumerate the mixtures and ingredients.

Mixtures		Ingredients	
Chewy	1	Seeds	1
Crunchy	2	Raisins	2
Nutty	3	Peanuts	3

Next we have a few given constants.

$$\begin{aligned} p_i &= \text{price per kg of mixture } i, \quad i = 1,2,3 \\ c_j &= \text{cost per kg of ingredient } j, \quad j = 1,2,3 \\ m_j &= \text{the maximum number of kgs of ingredient } j \\ &\quad \text{that can be delivered in a week,} \quad j = 1,2,3 \end{aligned}$$

These values are given above, so I will not recopy them here.

Finally I will define the following variables.

$$\begin{aligned} x_{ij} &= \text{percentage of ingredient } j \text{ in each kg of mixture } i \\ y_i &= \text{number of kilograms of mixture } i \text{ made each week} \end{aligned}$$

Before we go further, we need to make something very clear. There is no reference in the statement of the problem about the sales of these mixtures. I am going to assume that the demand for these snacks is so large that each mixture will sell out no matter how much is made. Without this assumption, we cannot write down the profit for the week.

Note that given the variables above, the cost to buy the ingredients in a week is given by

$$\text{Cost} = \sum_{i=1}^{3} y_i \sum_{j=1}^{3} c_j x_{ij}$$

while the sales revenue, with the above assumption, is

$$\text{Revenue} = \sum_{i=1}^{3} y_i p_i.$$

Then the profit for the week is given by

$$\text{Profit} = \text{Revenue} - \text{Cost} = \sum_{i=1}^{3} y_i \left(p_i - \sum_{j=1}^{3} c_j x_{ij} \right).$$

This is our objective function. We hope to maximize this.

Of course, there are several constraints to consider. First our suppliers have a limit to the amount of ingredients they can deliver to us each week.

$$m_j \geq \sum_{i=1}^{3} y_i x_{ij}, \quad j = 1, 2, 3$$

The next set of constraints come from the recipe prescriptions given in the table above.

$$\begin{array}{lll} x_{11} = 0 & x_{12} \geq 0.6 & x_{13} \leq 0.2 \\ x_{21} \geq 0.6 & x_{22} = 0 & x_{23} \geq 0.1 \\ x_{31} \leq 0.2 & x_{32} = 0 & x_{33} \geq 0.6 \end{array}$$

The last set of constraints are reasonability constraints. A percentage is a number between 0 and 1. The percentages must sum to 1 for each mixture. We cannot make a negative number of kilograms of a mixture.

$$\begin{aligned} x_{ij} &\geq 0 \quad \forall i,\ j \\ x_{ij} &\leq 1 \quad \forall i,\ j \\ \sum_{j=1}^{3} x_{ij} &= 1 \quad \forall i \\ y_i &\geq 0 \quad \forall i \end{aligned}$$

Exercise 2 *Use Fourier-Motzkin method to solve the following linear program.*

$$\begin{array}{ll} \min & 2x_1 - x_2 \\ \text{s.t.} & -x_1 + x_2 \leq 3 \\ & x_1 + x_2 \leq 5 \\ & x_1 \geq 0 \end{array}$$

We introduce a new variable $x_0 = 2x_1 - x_2$. We carry out the Fourier-Motzkin method by first eliminating x_2.

$$\left.\begin{aligned} 2x_1 - x_2 &= x_0 \\ -x_1 + x_2 &\leq 3 \\ x_1 + x_2 &\leq 5 \\ x_1 &\geq 0 \end{aligned}\right\} \Longrightarrow \left.\begin{aligned} &x_2 = 2x_1 - x_0 \\ &x_2 \leq 3 + x_1 \\ &x_2 \leq 5 - x_1 \\ &x_1 \geq 0 \end{aligned}\right\} \Longrightarrow \left.\begin{aligned} 2x_1 - x_0 &\leq 3 + x_1 \\ 2x_1 - x_0 &\leq 5 - x_1 \\ x_1 &\geq 0 \end{aligned}\right\} \Longrightarrow \begin{aligned} &x_1 \leq 3 + x_0 \\ &3x_1 \leq 5 + x_0 \\ &x_1 \geq 0 \end{aligned}$$

We carry forward and project onto x_1.

$$\left.\begin{aligned} &x_1 \leq 3 + x_0 \\ &3x_1 \leq 5 + x_0 \\ &x_1 \geq 0 \end{aligned}\right\} \Longrightarrow \left.\begin{aligned} &0 \leq 3 + x_0 \\ &0 \leq \tfrac{1}{3}(5 + x_0) \end{aligned}\right\} \Longrightarrow \left.\begin{aligned} -3 &\leq x_0 \\ -5 &\leq x_0 \end{aligned}\right\} \Longrightarrow -3 \leq x_0$$

Since we want to minimize x_0, the solution is $x_0 = -3$.

Now that we have x_0, we can back up and find x_1.

$$\left.\begin{aligned} &x_1 \leq 3 + x_0 \\ &x_1 \geq 0 \end{aligned}\right\} \Longrightarrow \left.\begin{aligned} &x_1 \leq 0 \\ &x_1 \geq 0 \end{aligned}\right\} \Longrightarrow x_1 = 0$$

Finally we can compute x_2.

$$x_2 = 2x_1 - x_0 = 0 - (-3) = 3.$$

So the objective function has a minimum value of -3 at $(x_1, x_2) = (0, 3)$.

Exercise 3 *For each of the following sets, check if it is (1) a subspace, (2) an affine set, (3) a convex set, or (4) a cone. Justify your answers. (Notice that a subspace is both convex and affine.)*

First note that a subpace, which is closed under all scalar multiplication, is also a cone, which is closed under positive scalar multiplication. This is in addition to the fact noted above that a subspace is also convex and affine.

S is a subspace when $c \perp Null(A)$ and $b=0$

(a) $S = \{x \mid Ax = b, \quad c^\top x \geq 0\}$. S is affine when $c \perp Null(A)$ and $c^T x_0 \geq 0$ for some x_0 satisfying $Ax_0 = b$

First, this is not a subspace because it is not closed under scalar multiplication. Note that if $x \in S$, then $A(-x) = -b \neq b$ if $b \neq 0$. Similarly, this is not a cone because it is not closed under positive scalar multiplication since $A(2x) = 2b \neq b$ if $b \neq 0$.

Also this is not an affine space. To show this, we take the special case of $b = 0$. Note that $y = 0$ is a solution to $Ax = 0$ and $c^\top x \geq 0$. Now suppose that $x \in S$ is any other solution with $c^\top x > 0$. We take $\lambda = -1$. Then $c^\top(\lambda x + (1-\lambda)y) = c^\top(-x + 2y) = -c^\top x < 0$. So in general, this is not an affine space.

However, this is a convex set. This is a polyhedron so we know that this is convex, however we will demonstrate this here. Suppose that $x,\ y \in S$ and $\lambda \in [0, 1]$. Since $c^\top x \geq 0$, $c^\top y \geq 0$, $\lambda \geq 0$, and $1 - \lambda \geq 0$, we have that

$$c^\top(\lambda x + (1-\lambda)y) = \lambda c^\top x + (1-\lambda)c^\top y \geq 0.$$

Also

$$A(\lambda x + (1-\lambda)y) = \lambda Ax + (1-\lambda)Ay = \lambda b + (1-\lambda)b = b.$$

So S is closed under convex combination.

(b) $S = \{(x, y)^\top \mid y = Ax\}$.
Let $(x_1, y_1)^\top \in S$, $(x_2, y_2)^\top \in S$ and $a_1,\ a_2 \in \mathbb{R}$. Then note that

$$A(a_1x_1 + a_2x_2) = a_1Ax_1 + a_2Ax_2 = a_1y_1 + a_2y_2.$$

Thus $a_1(x_1, y_1)^\top + a_2(x_2, y_2)^\top \in S$. Since S is closed under linear combinations, S is a subspace. Then S is affine, convex, and a cone by the comments above.

(c) $S = \{x \in \mathbb{R}^2 \mid x_2 \geq e^{x_1}\}$.
We note that S is the epi-graph of $f(x) = e^x$, a convex function. So we know that S is a convex set.

Note that $(0, 0)^\top \notin S$, since $e^0 = 1 > 0$. This is enough to show that S is not a subspace and not a cone since $(0, 0)^\top$ is an element in all subspaces and all cones.

Finally we note that it is not affine. Note that $(0, e)^\top \in S$ because $e \geq 1$ and $(1, e)^\top \in S$ trivially. Consider the line that goes through these two points, $x_2 = e$. If S were affine, it would have to contain all points along this line. Note that $(2, e) \notin S$ since $e < e^2$. So S is not affine.

Exercise 4 *Let $S = a + T$ where $a \in \mathbb{R}^n$ and T is a subspace in $\mathbb{R}^n$. Show that $S = b + T$ for any vector $b \in S$.*

Fix $b \in S$. Since $S = a + T$, there is a $t_b \in T$ such that $b = a + t_b$. Let s be any element in S. Then there is a $t \in T$ with

$$s = a + t = a + t + (t_b - t_b) = (a + t_b) + (t - t_b) = b + \bar{t}$$

where $\bar{t} = t - t_b \in T$ because T is a subspace. This shows that every element of S can be written as $b + \bar{t}$, with $\bar{t} \in T$. That is, we can write $S = b + T$ for any chosen element $b \in S$.

Exercise 5 *Let y and z be optimal solutions of the linear program $\min\{c^\top x \mid Ax \leq b,\ x \geq 0\}$. Show that for any $\lambda \in [0, 1]$, the vector $\lambda y + (1 - \lambda)z$ is also optimal.*

First we note that $\lambda y + (1 - \lambda)z$ is a feasible solution. This is true because polyheadrons are convex sets, but we will show this from basic principles. Since $y \geq 0$, $z \geq 0$, $\lambda \geq 0$, and $1 - \lambda \geq 0$, we have that $\lambda y + (1 - \lambda)z \geq 0$. Next since $Ay \leq b$ and $Az \leq b$,

$$A\left(\lambda y + (1 - \lambda)z\right) = \lambda Ay + (1 - \lambda)Az \leq \lambda b + (1 - \lambda)b = b.$$

So $\lambda y + (1 - \lambda)z$ is a feasible solution.

Next we will show that $\lambda y + (1 - \lambda)z$ is optimal. We know that y and z are both optimal. That means that for all feasible x, $c^\top y = c^\top z \leq c^\top x$. Then

$$c^\top\left(\lambda y + (1 - \lambda)z\right) = \lambda c^\top y + (1 - \lambda)c^\top z = \lambda c^\top y + (1 - \lambda)c^\top y = c^\top y.$$

Thus for any $\lambda \in [0, 1]$, $\lambda y + (1 - \lambda)z$ is an optimal solution to the linear program.