Special thanks to Aubin for providing the excellent answer.

Exercise 1 *Use the enumeration method to solve the follwing linear program.*

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + 3x_2 \leq 12 \\ & x_1 - x_2 \leq 3 \\ & x_1 \geq 0 \\ & x_2 \geq 0. \end{aligned}$$

Since there are 4 contraints and 2 dimensions, we will need to find $\binom{4}{2} = 6$ basic solutions.

Basic Solution 1 We use $x_1 = 0$ and $x_2 = 0$ to get $\hat{x} = (0, 0)^\top$. This clearly meets the other contraints, since $0 \leq 12$ and $0 \leq 3$, so it is a vertex of the feasible region.

Basic Solution 2 We use $x_1 = 0$ and $x_1 + 3x_2 = 12$ to get $\hat{x} = (0, 4)^\top$. This does meet the other contraints. In particular, we note that $0 - 4 = -4 \leq 3$. So this too is a vertex.

Basic Solution 3 We use $x_1 = 0$ and $x_1 - x_2 = 3$ to get $\hat{x} = (0, -3)^\top$. This is clearly not a vertex because $-3 < 0$.

Basic Solution 4 We use $x_2 = 0$ and $x_1 + 3x_2 = 12$ to get $\hat{x} = (12, 0)^\top$. This is not a vertex because $12 - 0 = 12 > 3$.

Basic Solution 5 We use $x_2 = 0$ and $x_1 - x_2 = 3$ to get $\hat{x} = (3, 0)^\top$. This does meet the other constraints. In particular we note that $3 + 3 \cdot 0 = 3 \leq 12$. This is another vertex.

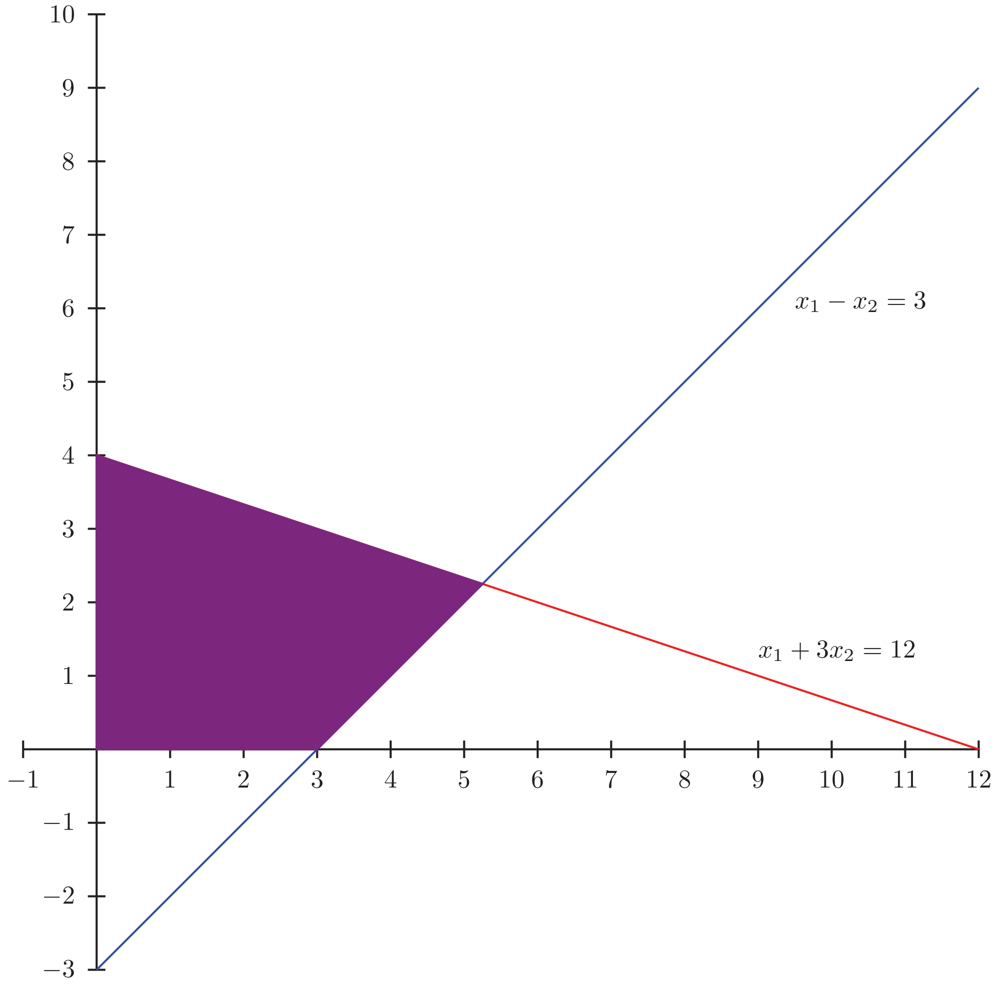
Basic Solution 6 We use $x_1 - x_2 = 3$ and $x_1 + 3x_2 = 12$. Here is the first time that the solution is not obvious. So for fun, we write out the steps we used in order to solve this system on linear equations.

$$\left.\begin{aligned} x_1 + 3x_2 &= 12 \\ -x_1 + x_2 &= -3 \end{aligned}\right\} \implies 4x_2 = 9 \implies x_2 = \frac{9}{4} = 2.25.$$

Similarly,

$$\left.\begin{aligned} x_1 + 3x_2 &= 12 \\ 3x_1 - 3x_2 &= 9 \end{aligned}\right\} \implies \quad 4x_1 = 21 \quad \implies \quad x_1 = \frac{21}{4} = 5.25.$$

Since these are both nonnegative, $\hat{x} = (5.25, 2.25)^\top$ is our fourth and final vertex.



The feasible region is colored purple in the image above. We can see that our four calculated vertices are indeed the four vertices of the region. It remains to calculate the objective function at these four points. Recall that the objective function is $x_1 + 2x_2$. Then we have

Vertex	Value
$(0, 0)$	0
$(0, 4)$	8
$(3, 0)$	3
$(5.25, 2.25)$	9.75

Since we are looking to maximize the objective function, the final row in the table above holds the solution. The maximum of 9.75 is achieved at $(5.25, 2.25)$.

Exercise 2 *Transform the following general linear program to (1) standard form and (2) canonical form, with effort to reduce the side of the resulting problems.*

$$\begin{aligned} \max \quad & 3x_1 - 4x_2 - x_3 + 6 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 0 \\ & x_1 - 2x_2 - x_3 \geq 4 \\ & x_3 \geq -3 \\ & x_1 + x_2 - x_3 = 5 \end{aligned}$$

First for canonical or standard form, we want to minimize our objective function. After dismissing the extraneous constant term, we want

$$\min \quad -3x_1 + 4x_2 + x_3$$

Now we note that there are several free variables in this linear program. We begin by removing x_1, a free variable, by using the first contraint. Now we have

$$x_1 = -x_2 - x_3.$$

Then our LP becomes

$$\begin{aligned} \min \quad & 7x_2 + 4x_3 \\ \text{s.t.} \quad & -3x_2 - 2x_3 \geq 4 \\ & x_3 \geq -3 \\ & -2x_3 = 5 \end{aligned}$$

At this point, we could note that x_3 must equal $-5/2$. Luckily, this meets the contraint that $x_3 \geq -3$. Knowing this, we can eliminate x_3 altogether to get the following LP.

$$\begin{aligned} \min \quad & x_2 \\ \text{s.t.} \quad & -3x_2 \geq -1 \end{aligned}$$

We can solve this by inspection. The feasible region is unbounded and there is no optimal solution. The above formulation is nice in that it certainly has a minimum number of contraints. However, this is not in either standard or canonical form.

We write $x_2 = x_2^p - x_2^n$, where $x_2^p = \max(0, x_2)$ is the postive part of of x_2, and $x_2^n = \max(0, -x_2)$ is the negative part of of x_2. Then our LP becomes

$$\begin{aligned} \min \quad & x_2^p - x_2^n \\ \text{s.t.} \quad & -3x_2^p + 3x_2^n \geq -1 \\ & x_2^p \geq 0 \\ & x_2^n \geq 0 \end{aligned}$$

This LP is in canonical form.

Finally, we may use a slack variable s to put this into standard form.

$$\begin{aligned} \min \quad & x_2^p - x_2^n \\ \text{s.t.} \quad & -3x_2^p + 3x_2^n - s = -1 \\ & x_2^p \geq 0 \\ & x_2^n \geq 0 \\ & s \geq 0 \end{aligned}$$

Exercise 3 *Consider a set P described by linear inequality contraints, that is*

$$P = \left\{ x \mid a_i^\top x \leq b_i, \quad i = 1, \ldots, m \right\}.$$

A ball with center y and radius r is the set of all points within Euclidean distance r from y. Formulate a linear program with can find the largest ball entirely contained in the set P.

Let $y \in \mathbb{R}^n$ be any point such that $a^\top y \leq b$ for some $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$. That is, y is a point in the half space "below" the hyperplane $H = \{x \mid a^\top x = b\}$. Since a is the normal vector of the hyperplane, the vector $v = \frac{\|a\|}{a}$ is the unit vector that points directly from y to the hyperplane.

The distance from y to the hyperplane is $d(y, H) = \min\{d \geq 0 \mid a^\top(y + dv) = b\}$. Then the ball centered at y with radius d will be tangent to H at $y + dv$. This gives us two facts. Firstly, if $\bar{d} < d$, then $a^\top(y + \bar{d}v) \leq b$. Secondly, if w is any direction other than v, then $a^\top(y + dw) \leq b$. We conclude that if $r \geq 0$ with $a^\top(y + rv) \leq b$, then every point $x \in B(y, r)$ satisfies $a_i^\top x \leq b_i$. So to ensure that the ball remains in the halfspace, we only need to check one point.

Knowing all of this, the linear program is easy to arrange. Define $v_i = \frac{\|a_i\|}{a_i}$ for all $i = 1, \ldots, m$.

$$\begin{aligned} \max \quad & r \\ \text{s.t.} \quad & r \geq 0 \\ & y \in P \\ & y + rv_i \in P \quad \forall i = 1, \ldots, m \end{aligned}$$

We can write this also as the following possibly more explicit form.

$$\begin{aligned} \max \quad & r \\ \text{s.t.} \quad & r \geq 0 \\ & a_i^\top y \leq b_i, \quad \forall i = 1, \ldots, m \\ & a_i^\top (y + rv_i) \leq b_i, \quad \forall i = 1, \ldots, m \end{aligned}$$

It is perhaps clearer now that the contraint $y \in P$ is extraneous given that we restrict $r \geq 0$. If there is an $r \geq 0$ with $a_i^\top(y + rv_i) \leq b$, then by the comments above, we must have $a_i^\top y \leq b_i$. Also by the comments above, the LP constrains the ball to lie in the correct halfspace for each of the m hyperplanes. That is, the resulting ball $B(y, r)$ will be contained in P.