Optimization
Homework 10

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Exercise 1 *Use duality to check if $x = (1, 0, 1, 0)$ is an optimal solution of the following linear program.*

$$\begin{array}{ll} \min & -x_1 + 2x_2 - x_3 - x_4 \\ \text{s.t.} & x_1 + x_2 - x_3 + 2x_4 \geq -2 \\ & x_1 + 2x_2 - x_3 + x_4 = 0 \\ & -x_1 - x_2 - x_3 - x_4 \geq -2 \\ & x_1,\ x_2,\ x_3 \geq 0 \\ & x_4 \text{ free} \end{array}$$

First we do a bit of work to reduce the problem. Note x_4 is free and is a term in the equality constraint. So we replace x_4 with $x_4 = -x_1 - 2x_2 + x_3$ everywhere in the problem.

$$\begin{array}{ll} \min & 4x_2 - 2x_3 \\ \text{s.t.} & x_1 + 3x_2 - x_3 \leq 2 \\ & - x_2 + 2x_3 \leq 2 \\ & x_1,\ x_2,\ x_3 \geq 0 \end{array}$$

We notice an interesting feature of this linear program. The variable x_1 appears only in the first constraint. Suppose we have some feasible point $\hat{x}$ with $\hat{x}_1 + 3\hat{x}_2 - \hat{x}_3 < 2$. We know that $\hat{x}_1 \geq 0$ because it is feasible. We can increase $\hat{x}_1$ without changing this fact. So we increase $\hat{x}$ to $\hat{x} + \delta > 0$ so that $\hat{x}_1 + \delta + 3\hat{x}_2 - \hat{x}_3 = 2$ without affecting any other constraints or the value of the objective function. That is, x_1 acts as a slack variable in this problem. We will write $\bar{x}_1$ to represent $x_1 + \delta$ to emphasize the special nature of this variable. For the second constraint, we do need to introduce a separate slack variable to put this into standard form. Then the problem becomes

$$\begin{array}{ll} \min & 4x_2 - 2x_3 \\ \text{s.t.} & \bar{x}_1 + 3x_2 - x_3 = 2 \\ & - x_2 + 2x_3 + u = 2 \\ & \bar{x}_1,\ x_2,\ x_3,\ u \geq 0 \end{array}$$

So the point in question, $(1, 0, 1, 0)$ has $\delta = 2$ and $u = 0$, so that $\bar{x}_1 = 1 + 2 = 3$. We can see this when we manipulate the tableau corresponding to this LP.

$$T_0 = \left[\begin{array}{c|cccc} 0 & 0 & 4 & -2 & 0 \\ \hline 2 & 1 & 3 & -1 & 0 \\ 2 & 0 & -1 & 2 & 1 \end{array}\right]$$

$$T_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0.5 \\ 0 & 0 & 0.5 \end{bmatrix} \; T_0 = \left[\begin{array}{c|cccc} 2 & 0 & 3 & 0 & 1 \\ \hline 3 & 1 & 2.5 & 0 & 0.5 \\ 1 & 0 & -0.5 & 1 & 0.5 \end{array}\right]$$

This tableau corresponds to $\bar{x}_1 = x_1 + \delta = 3$, $x_2 = 0$, $x_3 = 1$, and $u = 0$. This then corresponds to a parameterized family of points to the original problem; $(x_1, x_2, x_3, x_4) = (3 - \delta, 0, 1, \delta - 2)$ for all $0 \leq \delta \leq 3$.

But all of that was an aside. We want to use duality in this problem. First we write down the dual problem for reference.

$$\begin{aligned} \max \quad & 2w_1 + 2w_2 \\ \text{s.t.} \quad & w_1 \leq 0 \\ & 3w_1 - w_2 \leq 4 \\ & -w_1 + 2w_2 \leq -2 \\ & w_1,\ w_2 \leq 0 \end{aligned}$$

We want x_1 and x_3 to be basic variables. Take

$$B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad B^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}.$$

Since $b = (2,\ 2)^\top$, then

$$B^{-1}b = \frac{1}{2}\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

which is what we got with the tableau manipulations above. Also define

$$w = B^{-\top}c_B = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

The cost at this w is $2w_1 + 2w_2 = -2$, which is the same as the primal cost of $(1, 0, 1, 0)$ by construction. Next we note that w is feasible for the dual problem. Clearly $w \leq 0$. Moreover, we note

$$\begin{aligned} 3w_1 - w_2 &= 1 \leq 4 \\ -w_1 + 2w_2 &= -2 \leq -2 \end{aligned}$$

Thus w is feasible for the dual problem with the same dual objective function value as the primal objective function value at $(1, 0, 1, 0)$. Then the weak duality theorem tells us that w is optimal for the dual and $(1, 0, 1, 0)$ is optimal for the primal problem. By the comments above, we know that this is not the unique optimal solution to the primal problem, but only one optimal solution in a parameterized set of optimal solutions.

Exercise 2 *It is known that an optimal solution to the following linear program is of the form* $(0, \alpha, \beta)$ *with* $\alpha > 0$ *and* $\beta > 0$. *Without using any simplex pivots, solve the problem and its dual.*

$$\begin{aligned} \min \quad & -12x_1 + 10x_2 + 2x_3 \\ \text{s.t.} \quad & -4x_1 + x_2 - 8x_3 \geq 1 \\ & -x_1 + x_2 + 12x_3 \geq 3 \\ & x_1,\ x_2,\ x_3 \geq 0 \end{aligned}$$

We can put this LP into standard form by the introduction of two slack variables. Having done that, we want x_2 and x_3 to be the basic variables. Let

$$B = \begin{bmatrix} 1 & -8 \\ 1 & 12 \end{bmatrix} \quad \text{and} \quad B^{-1} = \frac{1}{20}\begin{bmatrix} 12 & 8 \\ -1 & 1 \end{bmatrix}.$$

Let $b = (1,\ 3)^\top$. Then

$$B^{-1}b = \frac{1}{20}\begin{bmatrix} 12 & 8 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 18 \\ 1 \end{bmatrix}.$$

We take $x = (0, 1.8, 0.1)^\top$. Then x has objective function value

$$-12x_1 + 10x_2 + 2x_3 = 18 + \frac{2}{10} = \frac{182}{10}.$$

Also note that x is feasible for the linear program. Clearly $x \geq 0$, and x meets the other two constraints with equality.

$$\begin{aligned} -4x_1 + x_2 - 8x_3 &= \frac{18}{10} - \frac{8}{10} = 1 \\ -x_1 + x_2 + 12x_3 &= \frac{18}{10} + \frac{12}{10} = 3. \end{aligned}$$

Now that we have the solution to the primal problem, the corresponding point for the dual problem is easy to compute. First we write down the dual problem for reference.

$$\begin{aligned} \max \quad & w_1 + 3w_2 \\ \text{s.t.} \quad & -4w_1 - w_2 \leq -12 \\ & w_1 + w_2 \leq 10 \\ & -8w_1 + 12w_2 \leq 2 \\ & w_1,\ w_2 \geq 0 \end{aligned}$$

We define

$$w = B^{-\top}c_B = \frac{1}{20}\begin{bmatrix} 12 & -1 \\ 8 & 1 \end{bmatrix}\begin{bmatrix} 10 \\ 2 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 59 \\ 41 \end{bmatrix}.$$

The objective function value for this w is

$$w_1 + 3w_2 = \frac{59}{10} + 3 \cdot \frac{41}{10} = \frac{182}{10},$$

which is the same as the cost at x for the primal problem, by construction. We finally note that this w is feasible for the dual problem. Clearly, $w \geq 0$ and also

$$\begin{aligned} -4w_1 - w_2 &= -4 \cdot \frac{59}{10} - \frac{41}{10} = -19.5 < -12 \\ w_1 + w_2 &= \frac{59}{10} + \frac{41}{10} = 10 \\ -8w_1 + 12w_2 &= -8 \cdot \frac{59}{10} + 12 \cdot \frac{41}{10} = 2. \end{aligned}$$

Because w is feasible for the dual with the same optimal cost as the solution to the primal problem above, the weak duality theorem tells us that they both are optimal solutions.

Exercise 3 *Use Farkas' Lemma to prove the following theorem of alternatives. If A is an $m \times n$ matrix and $b \in \mathbb{R}^m$, exactly one of the following two systems has a solution.*

$$\text{(1) } Ax = b \quad \text{(2) } A^\top w = 0,\ b^\top w = 1.$$

It is easiest to prove this directly. First suppose that there is an $x \in \mathbb{R}^n$ such that $Ax = b$. Further suppose that $A^\top w = 0$. Then

$$b^\top w = (Ax)^\top w = x^\top A^\top w = x^\top \mathbf{0} = 0 \neq 1.$$

So if (1) has a solution then (2) does not have a solution.

Next suppose that there is a $w \in \mathbb{R}^m$ such that $A^\top w = 0$ and $b^\top w = 1$. Let $x \in \mathbb{R}^n$. Then

$$(Ax)^\top w = x^\top A^\top w = x^\top \mathbf{0} = 0.$$

Since $b^\top w = 1$, we cannot have $Ax = b$ for any $x \in \mathbb{R}^n$. To spell this out, if $Ax = b$ then

$$0 = (Ax)^\top w = b^\top w = 1.$$

So if (2) has a solution then (1) does not have a solution.

Alternatively, we can prove this by way of Farkas' Lemma. Recall that Farkas says

$$(\exists x \geq 0 \ \text{ s.t. } \ Ax = b) \iff \left(b^\top w \leq 0 \ \forall w \text{ s.t. } \ A^\top w \leq 0\right)$$

First suppose that there is an x such that $Ax = b$. We write $x = x_+ - x_-$, where $x_+ \geq 0$ and $x_- \geq 0$. Define $b_+ = Ax_+$ and $b_- = -Ax_-$. Then

$$b = Ax = Ax_+ - Ax_- = b_+ + b_-.$$

Then by Farkas, we know two things.

$$b_+^\top w \leq 0 \ \forall w \text{ s.t. } \ A^\top w \leq 0 \qquad \text{and} \qquad b_-^\top w \leq 0 \ \forall w \text{ s.t. } \ A^\top w \geq 0$$

Putting these together

$$b^\top w = b_+^\top w + b_-^\top w \leq 0 \ \forall w \text{ s.t. } \ A^\top w = 0.$$

In particular, if $A^\top w = 0$ then we cannot have $b^\top w = 1$. Thus if (1) has a solution, then (2) does not have a solution.

Now suppose that (2) has a solution. Then we have a w such that $A^\top w = 0$ and $b^\top w = 1$. For a proof by contradiction, we suppose that there is an x such that $Ax = b$. Then the argument proceeds as above and we reach a contradiction. Thus if (2) has a solution, then (1) does not have a solution.

So while we can use Farka's Lemma to prove this statement, it is easier to prove it directly without reference to Farkas.

Exercise 4 *Find all the values of α and β such that the following linear program has an optimal solution with basic variables x_1 and x_4.*

$$
\begin{aligned}
\min \quad & x_1 + \beta x_2 - x_3 - x_4 \\
\text{s.t.} \quad & x_1 + \qquad \beta x_3 + x_4 = 4 + \alpha \\
& \qquad x_2 + \ x_3 + x_4 = 2 + 2\alpha \\
& x_i \geq 0 \quad \forall i.
\end{aligned}
$$

Let

$$
A = \begin{bmatrix} 1 & 0 & \beta & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 4+\alpha \\ 2+2\alpha \end{bmatrix} \qquad \text{and} \qquad c = \begin{bmatrix} 1 \\ \beta \\ -1 \\ -1 \end{bmatrix}.
$$

Then we are trying to minimize $c^\top x$ subject to $Ax = b$ and $x \geq 0$. If we want to take x_1 and x_4 as basic variables, we take

$$
B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad B^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}
$$

Note

$$
\begin{aligned}
B^{-1}A &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \beta & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & \beta-1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \\
B^{-1}b &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4+\alpha \\ 2+2\alpha \end{bmatrix} = \begin{bmatrix} 2-\alpha \\ 2+2\alpha \end{bmatrix} \\
c^\top - c_B^\top B^{-1}A &= \begin{bmatrix} 1 & \beta & -1 & -1 \end{bmatrix} - \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & \beta-1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & \beta & -1 & -1 \end{bmatrix} - \begin{bmatrix} 1 & -2 & \beta-2 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 2+\beta & 1-\beta & 0 \end{bmatrix}
\end{aligned}
$$

For this to be a solution, we need $B^{-1}b \geq 0$ and we need all the reduced costs to be nonnegative. Then we must have

$$
\left.\begin{aligned}
2-\alpha &\geq 0 \\
2+2\alpha &\geq 0 \\
2+\beta &\geq 0 \\
1-\beta &\geq 0
\end{aligned}\right\} \implies \begin{aligned} -1 \leq \alpha \leq 2 \\ -2 \leq \beta \leq 1 \end{aligned}
$$