Asymptotic Analysis for Optimal Investment and Consumption with Transaction Costs with Two Futures Contracts

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Abstract

We consider an agent who seeks to optimally invest and consume in the presence of proportional transaction costs. The agent can invest in two types of futures contracts, modeled as two correlated arithmetic Brownian motions, and in a money market account with constant rate of interest. She may also consume and get utility $U(c) \stackrel{\triangle}{=} \frac{c^p}{p},\ c \geq 0$, where $p \in (0,1)$ and c is the rate of consumption. The agent can control the rate of consumption and influence the evolution of wealth by controlling the number of futures contracts held. Proportional transaction costs $\lambda_i = \alpha_i \lambda$ are charged for every trade in futures contracts of type $i,\ i = 1,2$. All consumption is done from the money market account. The agent wishes to maximize the expected discounted integral over $[0,\infty)$ of the utility of consumption. We compute an asymptotic expansion of the value function in powers of $\lambda^{\frac{1}{3}}$.

Contents

1 Introduction

We consider the problem of an agent seeking to optimally invest and consume in the presence of proportional transaction costs. The agent can trade in two kinds of futures contracts, driven by two correlated Brownian motions. Our work considers futures contracts, rather then stocks, because it is mathematically simpler, but the formulation and expected solutions of the two problems are similar. Since a futures contract is marked to market every day, its value is always zero. The futures price is used to determine the cash flow received by someone who holds a futures contract. In particular, the person holding a futures contract receives the differences in the futures prices over time. Consequently, the absolute rather than the relative changes in the futures price determine the cash flow that accrues from holding the contract. Thus we model the futures price of the contract of type i by assuming that it is an arithmetic Brownian motion with drift and volatility $\mu_i, \sigma_i > 0$. The two Brownian motions are correlated, with correlation coefficient ρ. The agent can also hold cash in the money market account, which grows at a constant interest rate r, and she may also consume. The agent pays proportional transaction cost $\lambda_i = \alpha_i \lambda$ for trading futures contract of type i, with α_i a positive constant. All the transaction costs and the consumption are done from the money market account. The agent wishes to maximize the expected discounted integral over $[0, \infty)$ of the utility of consumption, with the utility function given as $U(c) \overset{\triangle}{=} \frac{c^p}{p}$, $c \geq 0$, where $p \in (0,1)$. This generalizes the work of Janeček & Shreve [31] by allowing the investor to trade in more then one risky asset. The difficulty and interest in the generalized problem arises because of possible correlation between the two risky assets. Without loss of generality, we may assume that $\alpha_2\mu_1 \geq \alpha_1\mu_2$. For the sake of brevity, we choose not to present the case $\alpha_2\mu_1 = \alpha_1\mu_2$ here, as it can be solved using similar technique. Thus we assume that $\alpha_2\mu_1 > \alpha_1\mu_2$. We further restrict the parameter ρ so that $-\frac{\alpha_2\sigma_1}{\alpha_1\sigma_2} < \rho < \frac{\alpha_2\sigma_1}{\alpha_1\sigma_2}$. We note that this restriction is not symmetrical in permutation of indices, as we have already assumed that $\alpha_2\mu_1 > \alpha_1\mu_2$.

The main theorem of this paper determines the leading term in the asymptotic expansion of the loss of the value function due to transaction costs. The proof is divided into two cases: the first case, when the two Brownian motions are independent; and the second case, when they are correlated. The second case is complicated by the fact that the value functions is apparently not twice continuously differentiable.

To treat the first case, a heuristic argument is employed to estimate the value function. The argument assumes that the value function is twice continuously differentiable and that it has an expansion in powers of $\lambda^{\frac{1}{3}}$ inside a region that we refer to as the *"no-trade"* (NT) region, a region in which it is optimal not to trade. On the outside, the value function is defined by an appropriate trading strategy. By computing and equating the derivatives across the boundary of the NT region, it is possible to identify the leading coefficients in the power series expansion of the value function. It is then possible to develop a rigorous proof by using the coefficients obtained from the heuristic argument to construct viscosity sub- and supersolutions for the Hamilton-Jacobi-Bellman (HJB) equations associated with this problem. For the rigorous proof, none of the assumptions from the heuristic argument are used. We then use two classical theorems from the theory of viscosity solutions for HJB equations that are common to both cases. The first asserts that the value function is indeed a viscosity solution of the HJB equation, while the second, which is based on a PDE comparison principle, asserts that any viscosity solution lies between any two sub- and supersolutions to the HJB equation; see Crandall, Ishii & Lions [16]. And finally, as a corollary of the two theorems, we identify the first three terms, the $O(\lambda^0)$, the $O(\lambda^{1/3})$, and the $O(\lambda^{2/3})$ terms, in the asymptotic expansion of the value function because the constructed sub- and supersolutions agree in these terms, differing only in the $O(\lambda)$ term.

In the second case of correlated Brownian motions, the heuristic argument fails, apparently because the second derivative of the value functions is not continuous across the boundary of the NT region. This makes the problem with two types of futures contracts substantially different from the problem with only one risky asset. In particular it means that alternative methods must be developed to construct tight upper and lower bounds on the value function. The upper bound is then constructed by considering an auxiliary problem based on independent Brownian motions whose value function dominates the value function in the problem of interest. The lower bound is obtained as a subsolution to the HJB equation, but constructed by a method different from the one used in the first case. These two bounds agree up to and including $O(\lambda^{2/3})$ and thus identify the $O(\lambda^0)$, the $O(\lambda^{1/3})$, and the $O(\lambda^{2/3})$ terms.

In the case of zero transaction cost and one risky asset, the agent's optimal policy is to keep a constant ratio of number of futures contract to the total wealth, which we call the *Merton proportion*. See Merton [41] for a solution to a similar problem. When $\lambda > 0$, the optimal policy is to trade as soon as the position is sufficiently far away from the Merton proportion. More specifically, the agent's optimal policy is to maintain her position inside the NT region. If the agent's position is initially outside the NT region, she should immediately sell or buy future contracts in order to move to its boundary. The agent then will trade only when her position is on the boundary of the NT region, and only as much as necessary to keep it from exiting it, while no trading occurs in the interior of the region; see Davis & Norman [19].

Magill & Constantinides [40] were the first to introduce transaction cost into Merton's model. Their analysis, despite being heuristic, gives an insight into the optimal strategy and the existence of the NT region. A more rigorous analysis, though under restrictive conditions, was given by Davis & Norman [19]. Dumas & Luciano [23] studied the problem of maximizing the long-term expected growth rate with no intermediate consumption in the presence of transaction costs. They also identified the optimal trading strategy and the boundaries of the NT region. The viscosity solution approach to that problem was pioneered by Shreve & Soner [46].

It is well known that the value function is a viscosity solution to the Hamilton-Jacobi-Bellman (HJB) equation for the optimal control problem with transaction costs. In case of one risky asset, the equation is a partial differential equation in two variables, which due to a homogeneity property of the power utility functions considered here can be reduced to an equation in one variable. It is easily seen that the value function is convex, and it is typically assumed that it is twice continuously differential. This regularity over the boundary of the NT region of the value function is generally referred to as *the principle of smooth fit*, and the problem as a *free boundary problem*. Shreve & Soner [46] used viscosity solutions techniques to prove this regularity in case of one tradable stock. The principle of smooth fit is typically assumed to hold in heuristic derivations in similar problems, see for example Dewynne, Howison, Law & Lee [22], Atkinson & Al-Ali [1] and Atkinson & Mokkhavesa [3], however, we strongly suspect that the principle of smooth fit fails in our problem when $\rho \neq 0$.

Constantinides [11] numerically computed the effect of transaction costs on the value function for the problem with one risky asset, and observed that transaction costs have a "first-order effect on assets' demand" and a "second-order effect on equilibrium asset return." This effect has been made precise by introducing power expansions for the value function and the boundaries of the NT region in powers of λ, a useful and perhaps more informative approach for obtaining explicit results. It was used by Janeček & Shreve [31], who employed the theory of viscosity solutions to rigorously find the power expansions of the value function up to $O(\lambda)$ and of the boundaries of the NT region up to the $O(\lambda^{\frac{2}{3}})$ for the case of single stock and power utility function. Later, Janeček & Shreve [32] using probabilistic arguments proved a similar result but with one futures contract.

While the problem of optimal investment in the presence of transaction costs is important in its own right, it has further value in the the study of contingent claim pricing. This problem has received considerable attention but often leads to a trivial result: since it's usually not possible to price the claim by replicating, super-replication is used, in which case the cheapest strategy turns out to be simply buy-and-hold. See [9], [17], [21], [25], [33], [34], [36], [37], [39], [48]. An alternative strategy to the buy-and-hold method was proposed by Leland [38]. His idea is to strike a balance between hemorrhaging money while trading, because of transaction costs, and"hedge slippage". This method leads to a modified Black-Scholes equation; see, for example, [5], [6], [7], [10], [24], [28], [30], [49].

Yet another alternative for contingent claims pricing in the presence of transaction costs, was proposed by Hodges and Neuberger [29]. Their idea is to price an option so that a utility maximizer is indifferent between either having a certain initial capital for investment or else holding the option but having initial capital reduced by the price of the option. This produces both a price and a hedge, the latter being the difference in the optimal trading strategies in the problem without the option and the problem with the option. This utility-based option pricing is examined in [8], [12], [13], [20]. A formal asymptotic analysis of such an approach appears in Whalley & Wilmott [50]. They assume a power expansion for the value function and compute the leading terms of it for both the case of holding the option liability and the case without it. Their proof corresponds to the heuristic derivation section in this paper.

The general case of optimal investment problem with multiple risky assets has proved to be more challenging. Most results have a restrictive assumption that the risky assets are independent. Atkinson & Wilmott [4] formally calculated a power series expansion of the value function in a model with fixed transaction costs studied by Morton & Pliska [42]. Atkinson & Al-Ali [1] gave a heuristic argument using perturbation analysis to calculate the value function and the location of the boundaries of the NT region assuming a power utility function. They considered the investment and consumption problem with one or more independent risky assets, and found the location of the boundaries to the order of $\lambda^{\frac{2}{3}}$ and the value function to the order of $\lambda^{\frac{4}{3}}$. Atkinson & Mokkhavesa [3] generalized the previous result to any nice utility function. They also computed the location of the boundaries to the order of $\lambda^{\frac{2}{3}}$ and the value function to the order of $\lambda^{\frac{4}{3}}$ in all the cases of two or more risky assets. More recently, Atkinson & Ingpochai [2] used the same method to calculate the value function for dependent risky assets. Their calculation of the value function and of the location of the boundary of the NT region is done asymptotically both in λ and in ρ, the maximum correlation coefficient between any two assets. They found the location of the boundaries to the order of $\lambda^{\frac{2}{3}}$ and $O(\rho)$ and the value function to the order of $\lambda^{\frac{4}{3}}$ and $O(\rho)$ in all the cases of two or more risky assets. Dewynne, Howison, Law & Lee [22] heuristically found a time-independent policy in an optimal investment without consumption in a finite-horizon problem with multiple correlated stocks. Under the assumption that *the principle of smooth fit* holds and that the boundaries are symmetrical around the Merton proportion, they heuristically computed the asymptotic location of the boundaries of the NT region.

Numerical results provided by Muthuraman & Kumar [43] for the problem with multiple stocks show how the optimal boundaries change with varying correlation ρ. Their computational method solves the free boundary problem by converting it into a sequence of fixed boundary problems. The method's update step relies on the principle of smooth fit. The following papers also provide numerical treatments of similar problems with transaction costs: Gennotte & Jung [26] considered the problem of optimal investment with transaction costs with one risky asset with finite time horizon. Using numerical analysis, they graphed the NT region as a function of time. Among other things, they showed that the NT widens to infinity as time to maturity approaches zero, and that the NT narrows and converges to a constant width as time to maturity increases. Dai & Zhong [18] study a finite time horizon optimal investment and consumption problem with multiple risky assets, in which the investor's goal is to maximize the discounted utility of consumption and the terminal wealth. The authors use penalty method technique to numerically solve the optimal investment problem, and present convergence analysis.

Goodman & Ostrov [27] showed how the heuristic PDE approach typically used in the above papers is related to a duality argument. They also showed that the approximate steady state probability density of the optimal portfolio can be infinite in two obtuse corners of the NT region.

In our problem, in the case of independent futures contracts in Section 3 we perform a heuristic calculation. We assume that our value function is twice continuously differentiable and has a power expansion. We then construct a system of equations by computing and equating the derivatives of the value function across the NT region. By solving this system of equations we find the first six terms of the power expansion of the value function.

However, there is no solution to the system with $\rho \neq 0$. This causes us to believe that the principle of smooth fit fails, and we conjecture that the value function is not everywhere twice continuously differentiable along the boundaries of the NT region, specifically not in its two obtuse corners, where Goodman & Ostrov [27] found that the density function blows up.

2 Model Definition

The set-up of the model is similar to Janeček & Shreve [32]. An agent is given an initial position of x dollars in the money market and y_i futures contracts of type i, for $i = 1, 2$. Given a filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$, the agent must choose a policy consisting of five adapted processes $L_i(t), M_i(t),\ i = 1, 2$, and $C(t)$, where the consumption process $C(t)$ is nonnegative and integrable on each finite interval, and the processes $L_i(t), M_i(t)$ are nondecreasing and right-continuous with left limits satisfying $L_i(0-) = 0, M_i(0-) = 0$. They are interpreted as the cumulative number of futures contracts of type i bought and sold by time t respectively.

Let $X(t)$ denote the wealth invested in the money market and $Y_i(t)$ the number of futures contracts of type i owned at time t, with $X(0-) = x$, $Y_i(0-) = y_i$. We adopt an arithmetic Brownian motion model for the change in futures price of contract of type i,

$$F_i(t) = F_i(0) + \mu_i t + \sigma_i B_i(t), \tag{2.1}$$

where μ_i, σ_i are positive constants, and $\{(B_1(t), B_2(t)), t \geq 0\}$ are correlated Brownian motions, such that $< B_1(t), B_2(t) >= \rho t$. The agent's position evolves as

$$\begin{aligned} dX(t) &= \sum_{i=1}^{2} Y_i(t) dF_i(t) - \sum_{i=1}^{2} \lambda_i (dL_i(t) + dM_i(t)) \qquad (2.2) \\ &\quad + (rX(t) - C(t))dt \\ dY_i(t) &= dL_i(t) - dM_i(t). \qquad (2.3) \end{aligned}$$

Note that

$$X(0) = x - \sum_{i=1}^{2} \lambda_i (L_i(0) + M_i(0)), \quad Y_i(0) = y_i + L_i(0) - M_i(0), \quad i = 1, 2, \tag{2.4}$$

may differ from the initial position (y_1, y_2, x) because of a transaction at time 0.

We assume a constant interest rate $0 < r < \min\{\mu_1, \mu_2\}$. The constants $\lambda_i \in (0, 1)$ appearing in these equations account for proportional transaction costs for trading in futures contract of type i, which are paid from the money market account. We will be doing an asymptotic analysis in transaction costs, so we need to control the ratio of transaction costs $\frac{\lambda_1}{\lambda_2}$. We assume $\lambda_i = \alpha_i \lambda$, where α_1 and α_2 are positive constants. It follows that $\lambda_i = O(\lambda)$. We also restrict the correlation coefficient $-\frac{\alpha_2 \sigma_1}{\alpha_1 \sigma_2} < \rho < \frac{\alpha_2 \sigma_1}{\alpha_1 \sigma_2}$.

As long as $X(u-) > 0,\ 0 \leq u \leq t$, we may define

$$l_i(t) = \int_0^t \frac{dL_i(u)}{X(u-)},\ m_i(t) = \int_0^t \frac{dM_i(u)}{X(u-)},\ c(t) = \int_0^t \frac{dC(u)}{X(u)},$$

to get

$$X(0)=x-X(0-)\sum_{i=1}^{2}\lambda_i\left(l_i(0)+m_i(0)\right),\quad Y_i(0)=y_i+X(0-)\left(l_i(0)-m_i(0)\right),\quad i=1,2 \tag{2.5}$$

and rewrite (2.2)-(2.3) as

$$\begin{aligned} dX(t) &= \sum_{i=1}^{2}Y_i(t)dF_i(t)-X(t-)\sum_{i=1}^{2}\lambda_i(dl_i(t)+dm_i(t)) \\ &\quad +X(t)(r-c(t))dt \end{aligned} \tag{2.6}$$

$$dY_i(t) = X(t-)(dl_i(t)-dm_i(t)). \tag{2.7}$$

It follows that $l_i(t), m_i(t), c(t)$ are non-decreasing, right-continuous processes, with $l_i(0-)=m_i(0-)=0$. We define the *solvency region* to be the open set

$$\mathcal{S}_v=\left\{(y_1,y_2,x)|x-\lambda_1|y_1|-\lambda_2|y_2|>0\right\}. \tag{2.8}$$

We have

$$(y_1,y_2,x)\in\mathcal{S}_v\Rightarrow x>0. \tag{2.9}$$

Let (y_1,y_2,x) be in $\mathcal{S}_v$. Then the policy $(l_i,m_i,c)_{i=1,2}$ is admissible for (y_1,y_2,x) if $X(0-)=x, Y_i(0-)=y_i$ and $(Y_1(t),Y_2(t),X(t))$, given by (2.6), (2.7) is in $\overline{\mathcal{S}_v}$ for all $t\geq 0$. We denote by $\mathcal{A}(y_1,y_2,x)$ the set of all such policies. A similar definition can be given for admissibility of $(L_i,M_i,C)_{i=1,2}$. Note that $(l_i,m_i,c)_{i=1,2}$ is admissible for (y_1,y_2,x) if and only if $(L_i,M_i,C)_{i=1,2}$ is admissible for (y_1,y_2,x). We will abuse notation and also write $(L_i,M_i,C)_{i=1,2}\in\mathcal{A}(y_1,y_2,x)$.

Remark 2.1. If $(y_1,y_2,x)\in\partial\mathcal{S}_v$, the only admissible policy is to jump immediately to the origin and remain there. In particular, $c(t)=0$ must be identically zero for all $t\geq 0$.

To see why this is true, consider, for specificity, the case that $(y_1,y_2,x)\in\partial\mathcal{S}_v$ satisfies $x-\lambda_1y_1-\lambda_2y_2=0$, and $y_1,y_2\geq 0$. Let $(l_i,m_i,c)_{i=1,2}\in\mathcal{A}(y_1,y_2,x)$ be given, and note from (2.5) that $X(0)-\lambda_1Y_1(0)-\lambda_2Y_2(0)=-2X(0-)\sum_{i=1}^{2}\lambda_il_i(0)$, which is not allowed to be negative. Since $X(0-)>0$, it follows that $l_i(0)=0,\ i=1,2$, and $(Y_1(0),Y_2(0),X(0))\in\partial\mathcal{S}_v$. Furthermore, (2.6)-(2.7) show that

$$\begin{aligned} & d\left[e^{-rt}\left(X(t)-\lambda_1Y_1(t)-\lambda_2Y_2(t)\right)\right] \\ &\quad = e^{-rt}\left[\sum_{i=1}^{2}Y_i(t)dF_i(t)-2X(t-)\sum_{i=1}^{2}\lambda_il_i(t)-c(t)X(t)dt\right] \end{aligned} \tag{2.10}$$

Let

$$\tau\triangleq\inf\{t\geq 0; Y_1(t)\notin(0,y_1+1)\text{ or }Y_2(t)\notin(0,y_2+1)\},$$

and integrate (2.10) to obtain

$$\begin{aligned} 0 &\leq e^{-r\tau}\left(X(\tau) - \lambda_1 Y_1(\tau) - \lambda_2 Y_2(\tau)\right) \\ &= \int_0^{\tau} e^{-rt}\left[\sum_{i=1}^{2} Y_i(t) dF_t(t) - c(t)X(t)dt - 2X(t-)\sum_{i=1}^{2}\lambda_i dl_i(t)\right] \\ &\leq \int_0^{\tau} e^{-rt}\sum_{i=1}^{2} Y_i(t)\left(\mu_i dt + \sigma_i dB_i(t)\right). \end{aligned}$$

According to Girsanov's theorem, there is a probability measure, mutually absolutely continuous with respect to $\mathbb{P}$, under which $(\mu_1 t + \sigma_1 B_1(t), \mu_2 t + \sigma_2 B_2(t))$ for $0 \leq t \leq \tau$ is a two dimensional ρ–correlated Brownian motion. Under this measure, the nonnegative random variable

$$\int_0^{\tau} e^{-rt}\sum_{i=1}^{2} Y_i(t)\left(\mu_i dt + \sigma_i dB_i(t)\right)$$

has expectation zero, and so is almost surely zero. This implies that $\tau = 0$, almost surely, from which we conclude that since $l_i(0) = 0,\ i = 1, 2$, then either $Y_1(0) = 0$ or $Y_2(0) = 0$, and possibly both. If only $Y_1(0) = 0$, then repeating the above calculation with

$$\tilde{\tau} \triangleq \inf\{t \geq 0; Y_1(t) \notin (-1, y_1 + 1) \text{ or } Y_2(t) \notin (0, y_2 + 1)\},$$

we conclude again that $\tilde{\tau} = 0$. Since we know that $Y_1(0) = 0$ it follows that $Y_2(0) = 0$.

Similar computation can be done on other parts of the boundary of the solvency region $\mathcal{S}_v$.

We introduce the agent's *utility function* $U : (0, \infty) \mapsto \mathbb{R}$ defined for all $c \geq 0$ by $U(c) \triangleq \frac{c^p}{p}$. The parameter p is in $(0, 1)$.

Let $\beta > 0$ be a positive discount rate and define the value function

$$v(y_1, y_2, x) \triangleq \sup_{(l_i, m_i, c) \in \mathcal{A}(y_1, y_2, x)} E\left[\int_0^{\infty} e^{-\beta t} U(X(t)c(t))dt\right]. \tag{2.11}$$

Remark 2.2. It now follows from Remark 2.1 that the value function v is zero on the boundary of the solvency region, i.e., $v|_{\partial\mathcal{S}_v} = 0$. □

The problem with $\lambda = 0$ requires a different formulation, one in which the control processes are Y_1, Y_2 and C, and Y_1 andY_2 are not required to be finite variation processes. One can define the value function in this problem, and adapting the analysis of Merton [41], one can show that the optimal policy always keeps the proportion of number of futures contracts of type i to wealth constant and equal to

$$\theta_i = \frac{\mu_i \sigma_j - \rho \mu_j \sigma_i}{(1-p)\sigma_i^2 \sigma_j (1 - \rho^2)} \tag{2.12}$$

We call (θ_1, θ_2) the *Merton proportion.* For $\lambda = 0$, the value function is

$$v(y_1, y_2, x) = \frac{1}{p} A^{p-1} x^p, \tag{2.13}$$

where

$$A = \frac{\beta - rp}{1-p} - \frac{p}{1-p} \sum_{i=1}^{2} \mu_i \theta_i + \frac{p}{2} \left(\sigma_1^2 \theta_1^2 + \sigma_2^2 \theta_2^2 + 2\rho\sigma_1\sigma_2\theta_1\theta_2 \right) \tag{2.14}$$

The optimal consumption proportion is $c(t) = A$.

We assume throughout that $A > 0$, which is a necessary and sufficient condition that the value function for the problem with zero transaction cost is finite.

We introduce the *convex dual function* $\widetilde{U} : (0, \infty) \mapsto \mathbb{R}$ defined by

$$\widetilde{U}(\tilde{c}) \triangleq \sup_{c>0} \{U(c) - c\tilde{c}\} = \frac{1-p}{p} \tilde{c}^{-\frac{p}{1-p}}. \tag{2.15}$$

The supremum in (2.15) is attained by $c = I_p(\tilde{c})$, where

$$I_p(\tilde{c}) \triangleq \tilde{c}^{1/(p-1)} \qquad \forall \tilde{c} > 0 \tag{2.16}$$

is the inverse of the strictly decreasing function U'. We have from (2.15) that

$$\widetilde{U}(\tilde{c}) + c\tilde{c} - U(c) \geq 0 \qquad \forall c \geq 0, \tilde{c} > 0. \tag{2.17}$$

Use convexity of the function $c \mapsto \widetilde{U}(\tilde{c}) + c\tilde{c} - U(c)$ to get for any constant $k > 0$

$$\widetilde{U}(\tilde{c}) + c\tilde{c} - U(c) \geq \widetilde{U}(\tilde{c}) + k\tilde{c} - U(k) + (c-k)(\tilde{c} - U'(k)).$$

Together with (2.17) with k replacing c we conclude that

$$\widetilde{U}(\tilde{c}) + c\tilde{c} - U(c) \geq (c-k)(\tilde{c} - U'(k)) \qquad \forall c > 0. \tag{2.18}$$

Lemma 2.3. Assume $p < 1$, $p \neq 0$. For $a > 0$ and $b < a$ we have

$$\begin{aligned} \widetilde{U}(a-b) &= \frac{1-p}{p}\,(a-b) \left(a^{-\frac{1}{1-p}} + \frac{b}{1-p}\, a^{-\frac{2-p}{1-p}} + O\big(b^2\big) \right) \\ &= \frac{1-p}{p}\, a^{-\frac{p}{1-p}} + b\, a^{-\frac{1}{1-p}} + O\big(b^2\big). \end{aligned} \tag{2.19}$$

Proof: We write

$$\widetilde{U}(a-b) = \frac{1-p}{p} \big(a-b\big)^{-\frac{p}{1-p}} = \frac{1-p}{p} (a-b)\,(a-b)^{-\frac{1}{1-p}}.$$

A Taylor series expansion of the function $f(x) = (x-h)^{-\frac{1}{1-p}}$ yields $(a-b)^{-\frac{1}{1-p}} = a^{-\frac{1}{1-p}} + \frac{1}{1-p}\, b\, a^{-\frac{2-p}{1-p}} + O\big(b^2\big)$, and we get the desired result.

2.1 The Hamilton-Jacobi-Bellman (HJB) equation

The proof of the following classical theorem is deferred to the Appendix.

Theorem 2.4. The value function $v(y_1, y_2, x)$ defined by (2.11) is a viscosity solution of the following HJB equation

$$\min\left\{\mathcal{L}(v) - \widetilde{U}(v_x); \lambda_1 v_x - v_1; \lambda_2 v_x - v_2; \lambda_1 v_x + v_1; \lambda_2 v_x + v_2\right\} = 0 \tag{2.20}$$

on $\mathcal{S}_v$, where

$$\mathcal{L}(\psi) = \beta\psi - \frac{1}{2}\left(\sigma_1^2 y_1^2 + 2\rho\sigma_1\sigma_2 y_1 y_2 + \sigma_2^2 y_2^2\right)\psi_{xx} - \left(rx + \sum_{i=1}^{2}\mu_i y_i\right)\psi_x. \tag{2.21}$$

Intuitively, the solvency region is divided into nine regions. One where the second order operator $\mathcal{L}(v) - \widetilde{U}(v_x)$ is zero, i.e. it is optimally not to trade, while the other first order operators are all positive. Four, where only one first order operator is zero, i.e. it is optimally to make only one type of trade, and the other three first order operators and the second order operator are positive. For example, the region where $\lambda_i v_i - v_x$ is zero in the region corresponding to the region where it is optimal to buy futures contract of type i ($x \to y_i$), and the region where $\lambda_i v_x + v_i$ is zero is the region where it is optimal to sell futures contract of type i. Finally, there are four more regions, where two types of trades are optimal. These are the regions where two first order operators are zero, while all other are positive. For example, the region where it is optimal to buy futures contracts of both types, is the region where $\lambda_i v_i - v_x = 0,\ i = 1, 2$. We note, that it is never optimal to both buy and sell the same type of futures contracts, because of transaction costs begin deducted from the money market account.

Lemma 2.5. The value function v is homogeneous of degree p, i.e.,

$$v(\gamma y_1, \gamma y_2, \gamma x) = \gamma^p v(y_1, y_2, x) \tag{2.22}$$

for all $(y_1, y_2, x) \in \overline{\mathcal{S}_v}$ and $\gamma > 0$.

PROOF: This follows from the fact that $(c, l_i, m_i)_{i=1,2} \in \mathcal{A}(y_1, y_2, x)$ if and only if

$(\gamma c, \gamma l_i, \gamma m_i)_{i=1,2} \in \mathcal{A}(\gamma y_1, \gamma y_2, \gamma x)$. □

Consider the following reduction of variables, due to homogeneity of degree p of the value function v:

$$u(z_1, z_2) \triangleq v(1, z_1, z_2), \tag{2.23}$$

$$v(x, y_1, y_2) = x^p u\left(\frac{y_1}{x}, \frac{y_2}{x}\right). \tag{2.24}$$

We define the solvency region for the two-variables value function to be the open set

$$\mathcal{S}_u = \left\{(z_1, z_2) | 1 - \lambda_1 |z_1| - \lambda_2 |z_2| > 0\right\}. \tag{2.25}$$

Remark 2.6. It follows from Remark 2.2 and (2.23) that the value function u is zero on the boundary of the solvency region, i.e.,$u|_{\partial \mathcal{S}_u} = 0$. □

The proof of the following lemma, which follows from the classical Theorem 2.4, is also deferred to the Appendix.

Lemma 2.7. On $\mathcal{S}_u$, u is a viscosity solution of the HJB equation

$$\mathcal{H}(u) \triangleq \min\left\{(\mathcal{D} - \widetilde{\mathcal{U}})(u), \mathcal{B}_1(u), \mathcal{S}_1(u), \mathcal{B}_2(u), \mathcal{S}_2(u)\right\} = 0 \tag{2.26}$$

where

$$\begin{aligned}
\mathcal{D}(\psi) &= (1-p)\left[A + \frac{p}{2}\left[\sigma_1^2 (z_1 - \theta_1)^2 + \sigma_2^2 (z_2 - \theta_2)^2 + 2\rho\sigma_1\sigma_2 (z_1 - \theta_1)(z_2 - \theta_2)\right]\right]\psi \\
&\quad + \left[r + \sum_{i=1}^{2} \mu_i z_i - (1-p)(\sigma_1^2 z_1^2 + \sigma_2^2 z_2^2 + 2\rho\sigma_1\sigma_2 z_1 z_2)\right]\left[z_1\psi_1 + z_2\psi_2\right] \qquad (2.27) \\
&\quad - \frac{1}{2}\left[\sigma_1^2 z_1^2 + \sigma_2^2 z_2^2 + 2\rho\sigma_1\sigma_2 z_1 z_2\right]\left[z_1^2\psi_{11} + 2 z_1 z_2 \psi_{12} + z_2^2 \psi_{22}\right], \\
\widetilde{\mathcal{U}}(\psi) &= \widetilde{U}\left(p\psi - z_1\psi_1 - z_2\psi_2\right), \qquad (2.28) \\
\mathcal{B}_i(\psi) &= \lambda_i p\psi + (-1 - \lambda_i z_i)\psi_i - \lambda_i z_j \psi_j, \qquad (2.29) \\
\mathcal{S}_i(\psi) &= \lambda_i p\psi + (1 - \lambda_i z_i)\psi_i - \lambda_i z_j \psi_j. \qquad (2.30)
\end{aligned}$$

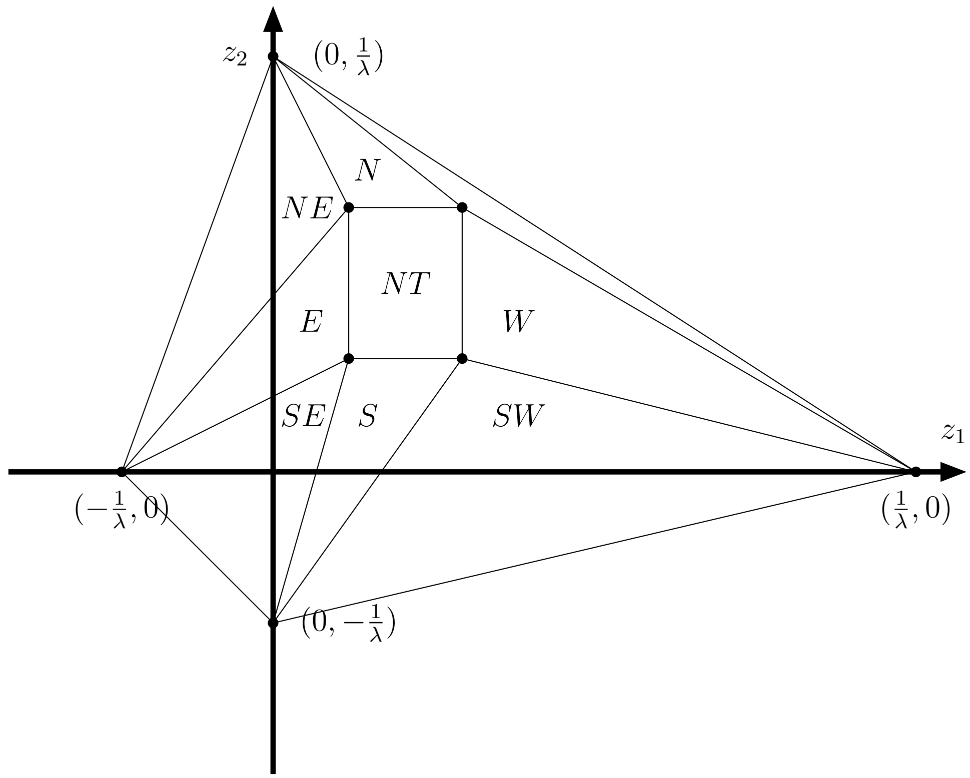
2.2 Partitioning the solvency region

Figure 1 shows the solvency region $\mathcal{S}_u$ in the variables (z_1, z_2), the region in which u is defined and satisfies (2.26) in the viscosity sense. According to (2.26), in the solvency region, we should have

$$\begin{aligned}
\mathcal{D}(u) - \widetilde{U}\left(pu - z_1 u_1 - z_2 u_2\right) &\geq 0, \qquad (2.31) \\
\lambda_1 pu + (-1 - \lambda_1 z_1)u_1 - \lambda_1 z_2 u_2 &\geq 0, \qquad (2.32) \\
\lambda_2 pu + (-1 - \lambda_2 z_2)u_2 - \lambda_2 z_1 u_1 &\geq 0, \qquad (2.33) \\
\lambda_1 pu + (1 - \lambda_1 z_1)u_1 - \lambda_1 z_2 u_2 &\geq 0, \qquad (2.34) \\
\lambda_2 pu + (1 - \lambda_2 z_2)u_2 - \lambda_2 z_1 u_1 &\geq 0. \qquad (2.35)
\end{aligned}$$

At each point in the solvency region, at least one of these inequalities should hold with equality. Indeed, we divide the solvency region into nine open regions, NT, N, S, E, W,

Figure 1: Partitioning of the Solvency Region



NE, NW, SE, and SW; see Figure 1. In particular, the solvency region corner $(-\frac{1}{\lambda}, 0) \notin W$, and $z_2 > 0$ for all $(z_1, z_2) \in W$.

The correspondences among the nine regions, the transactions that take place in each region, and the inequalities (2.31)–(2.35) that are equalities are shown in the following Table 1.

3 Heuristic Derivation for the case $\rho = 0$.

There is no explicit solution to (2.26) in the NT region, so in this section we do the next best thing - we derive several terms of a power series expansion of the value function by a heuristic method. Guided by the discussion in [45] and [31], we will assume that in this region $u(z_1, z_2)$ has an expansion in powers of $\lambda^{1/3}$, and we expect the coefficient of $\lambda^{1/3}$ to be zero. In order to work with this expansion, we need to also include the variables (z_1, z_2), and we do that using powers of $z_i - \theta_i$. In the next section we will use this heuristic expansion to build tight upper and lower bounds to the solution of (2.26). The assumptions presented below are only assumed throughout the heuristic section, and would not be used for the rigorous construction of the sub- and super-solutions presented in later sections.

Throughout this section we will assume the following:

Assumption 3.1.

(i) The value function function $u \in C^2(\overline{NT})$. The limits of its first- and second-order partial derivatives are defined on the boundaries of NT, and these limits are the partial derivatives at the boundaries computed along directions into NT.

(ii) The solvency region can be partitioned into nine regions, as in Figure 1. SWe assume that all the equalities from Table 1 hold on the boundaries of the appropriate regions. For example, the first-order partial derivatives of u satisfy (2.32) with equality on the western boundary of NT, satisfy (2.33) with equality on the southern boundary, etc.

(iii) For λ small enough, the NT region, contains the Merton proportion (θ_1, θ_2) and its size is of order $\lambda^{\frac{1}{3}}$, i.e. if $(z_1, z_2) \in NT$, then $z_i - \theta_i = O\left(\lambda^{\frac{1}{3}}\right)$. Moreover, assume that the size of the NT region can be written as a power expansion in terms of powers of $\lambda^{\frac{1}{3}}$.

(iv) Inside the NT region the value function u has a power series expansion in powers of $\lambda^{\frac{1}{3}}$ of the form

$$
\begin{aligned}
&u(z_1, z_2) \qquad (3.1)\\
&= \gamma_0 - \gamma_1\lambda^{\frac{1}{3}} - \gamma_2\lambda^{\frac{2}{3}} - \gamma_3\lambda - (\gamma_{4,1}(z_1-\theta_1) + \gamma_{4,2}(z_2-\theta_2))\lambda \\
&\quad -(\gamma_{5,1}(z_1-\theta_1)^2 + \gamma_{5,2}(z_1-\theta_1)(z_2-\theta_2) + \gamma_{5,3}(z_2-\theta_2)^2)\lambda^{\frac{2}{3}} \\
&\quad -(\gamma_{6,1}(z_1-\theta_1)^3 + \gamma_{6,2}(z_1-\theta_1)^2(z_2-\theta_2) + \gamma_{6,3}(z_1-\theta_1)(z_2-\theta_2)^2 + \gamma_{6,4}(z_2-\theta_2)^3)\lambda^{\frac{1}{3}} \\
&\quad -(\gamma_{7,1}(z_1-\theta_1)^4 + \gamma_{7,2}(z_1-\theta_1)^3(z_2-\theta_2) + \gamma_{7,3}(z_1-\theta_1)^2(z_2-\theta_2)^2 \\
&\qquad +\gamma_{7,4}(z_1-\theta_1)(z_2-\theta_2)^3 + \gamma_{7,5}(z_2-\theta_2)^4) + O(\lambda^{\frac{5}{3}})
\end{aligned}
$$

Table 1: Partitioning of the solvency Region

Region	Trading	Equality
NT	No trading	(2.31)
W	Buy futures contracts of type 1	(2.32)
SW	Buy futures contracts of types 1 and 2	(2.32), (2.33)
S	Buy futures contracts of type 2	(2.33)
SE	Buy futures contracts of type 2 and sell contracts of type 1	(2.33), (2.34)
E	Sell futures contracts of type 1	(2.34)
NE	Sell futures contracts of types 1 and 2	(2.34), (2.35)
N	Sell futures contracts of type 2	(2.35)
NW	Buy futures contracts of type 1 and sell contracts of type 2	(2.35), (2.32)

The dependency in z_1, z_2 is always through $z_i - \theta_i$, which inside the NT region is order $\lambda^{\frac{1}{3}}$. Note, that one could include the terms of order $\frac{1}{3}$, $\frac{2}{3}$ and 1 in λ that depend on z_1 and z_2 in the expansion (3.1). However, the heuristic calculations that follows would then show that these terms must be zero. We simplify the calculations by assuming these terms are zero.

(v) The NT region is a non-degenerate quadrilateral to the order of $lambda^{\frac{2}{3}}$ with vertices $(z_{1,NE}, z_{2,NE})$, $(z_{1,NW}, z_{2,NW})$, $(z_{1,SW}, z_{2,SW})$, and $(z_{1,SE}, z_{2,SE})$.
More specifically, in case of independent futures contracts, when $\rho = 0$, it's a rectangular.

It follows from Assumption 3.1 (iv) that inside the NT region

$$\begin{aligned}
&u_1(z_1, z_2) \\
&= -\gamma_{4,1}\lambda - (2\gamma_{5,1}(z_1 - \theta_1) + \gamma_{5,2}(z_2 - \theta_2))\lambda^{\frac{2}{3}} \\
&\quad -(3\gamma_{6,1}(z_1 - \theta_1)^2 + 2\gamma_{6,2}(z_1 - \theta_1)(z_2 - \theta_2) + \gamma_{6,3}(z_2 - \theta_2)^2)\lambda^{\frac{1}{3}} \\
&\quad -(4\gamma_{7,1}(z_1 - \theta_1)^3 + 3\gamma_{7,2}(z_1 - \theta_1)^2(z_2 - \theta_2) + 2\gamma_{7,3}(z_1 - \theta_1)(z_2 - \theta_2)^2 + \gamma_{7,4}(z_2 - \theta_2)^3) \\
&\quad +O(\lambda^{\frac{4}{3}}),
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&u_2(z_1, z_2) \\
&= -\gamma_{4,2}\lambda - (2\gamma_{5,3}(z_2 - \theta_2) + \gamma_{5,2}(z_1 - \theta_1))\lambda^{\frac{2}{3}} \\
&\quad -(3\gamma_{6,4}(z_2 - \theta_2)^2 + 2\gamma_{6,3}(z_2 - \theta_2)(z_1 - \theta_1) + \gamma_{6,2}(z_1 - \theta_1)^2)\lambda^{\frac{1}{3}} \\
&\quad -(4\gamma_{7,5}(z_2 - \theta_2)^3 + 3\gamma_{7,4}(z_2 - \theta_2)^2(z_1 - \theta_1) + 2\gamma_{7,3}(z_2 - \theta_2)(z_1 - \theta_1)^2 + \gamma_{7,2}(z_1 - \theta_1)^3) \\
&\quad +O(\lambda^{\frac{4}{3}}),
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
&u_{11}(z_1, z_2) \\
&= -2\gamma_{5,1}\lambda^{\frac{2}{3}} - (6\gamma_{6,1}(z_1 - \theta_1) + 2\gamma_{6,2}(z_2 - \theta_2))\lambda^{\frac{1}{3}} \\
&\quad -(12\gamma_{7,1}(z_1 - \theta_1)^2 + 6\gamma_{7,2}(z_1 - \theta_1)(z_2 - \theta_2) + 2\gamma_{7,3}(z_2 - \theta_2)^2) + O(\lambda),
\end{aligned} \tag{3.4}$$

$$\begin{aligned}
&u_{22}(z_1, z_2) \\
&= -2\gamma_{5,3}\lambda^{\frac{2}{3}} - (6\gamma_{6,4}(z_2 - \theta_2) + 2\gamma_{6,3}(z_1 - \theta_1))\lambda^{\frac{1}{3}} \\
&\quad -(12\gamma_{7,5}(z_2 - \theta_2)^2 + 6\gamma_{7,4}(z_2 - \theta_2)(z_1 - \theta_1) + 2\gamma_{7,3}(z_1 - \theta_1)^2) + O(\lambda),
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&u_{12}(z_1, z_2) \\
&= -\gamma_{5,2}\lambda^{\frac{2}{3}} - (2\gamma_{6,2}(z_1 - \theta_1) + 2\gamma_{6,3}(z_2 - \theta_2))\lambda^{\frac{1}{3}} \\
&\quad -(3\gamma_{7,2}(z_1 - \theta_1)^2 + 4\gamma_{7,3}(z_1 - \theta_1)(z_2 - \theta_2) + 3\gamma_{7,4}(z_2 - \theta_2)^2) + O(\lambda).
\end{aligned} \tag{3.6}$$

For $(z_1, z_2) \in NT$ we use (3.1)–(3.6) and substitute into (2.27), to get

$$
\begin{aligned}
&\mathcal{D}u(z_1, z_2) \qquad (3.7)\\
&= (1-p)A\gamma_0 - (1-p)A\gamma_1\lambda^{\frac{1}{3}} - (1-p)A\gamma_2\lambda^{\frac{2}{3}} \\
&\quad + \frac{p(1-p)}{2}\left[\sigma_1^2(z_1-\theta_1)^2 + \sigma_2^2(z_2-\theta_2)^2 + 2\rho\sigma_1\sigma_2(z_1-\theta_1)(z_2-\theta_2)\right]\gamma_0 \\
&\quad + \left[\sigma_1^2 z_1^2 + \sigma_2^2 z_2^2 + 2\rho\sigma_1\sigma_2 z_1 z_2\right] \\
&\qquad \times \Big[\gamma_{5,1}\lambda^{\frac{2}{3}} + (3\gamma_{6,1}(z_1-\theta_1) + \gamma_{6,2}(z_2-\theta_2))\lambda^{\frac{1}{3}} \\
&\qquad\quad + (6\gamma_{7,1}(z_1-\theta_1)^2 + 3\gamma_{7,2}(z_1-\theta_1)(z_2-\theta_2) + \gamma_{7,3}(z_2-\theta_2)^2)\Big] z_1^2 \\
&\quad + \left[\sigma_1^2 z_1^2 + \sigma_2^2 z_2^2 + 2\rho\sigma_1\sigma_2 z_1 z_2\right] \\
&\qquad \times \Big[\gamma_{5,2}\lambda^{\frac{2}{3}} + 2(\gamma_{6,2}(z_1-\theta_1) + \gamma_{6,3}(z_2-\theta_2))\lambda^{\frac{1}{3}} \\
&\qquad\quad + (3\gamma_{7,2}(z_1-\theta_1)^2 + 4\gamma_{7,3}(z_1-\theta_1)(z_2-\theta_2) + 3\gamma_{7,4}(z_2-\theta_2)^2)\Big] z_1 z_2 \\
&\quad + \left[\sigma_1^2 z_1^2 + \sigma_2^2 z_2^2 + 2\rho\sigma_1\sigma_2 z_1 z_2\right] \\
&\qquad \times \Big[\gamma_{5,3}\lambda^{\frac{2}{3}} + (3\gamma_{6,4}(z_2-\theta_2) + \gamma_{6,3}(z_1-\theta_1))\lambda^{\frac{1}{3}} \\
&\qquad\quad + (6\gamma_{7,5}(z_2-\theta_2)^2 + 3\gamma_{7,4}(z_2-\theta_2)(z_1-\theta_1) + \gamma_{7,3}(z_1-\theta_1)^2)\Big] z_2^2 + O(\lambda).
\end{aligned}
$$

Furthermore,

$$pu - (u_1 z_1 + u_2 z_2) = p\gamma_0 - p\gamma_1\lambda^{\frac{1}{3}} - p\gamma_2\lambda^{\frac{2}{3}} + O(\lambda). \qquad (3.8)$$

Setting $a = p\gamma_0$ and $b = p\gamma_1\lambda^{\frac{1}{3}} + p\gamma_2\lambda^{\frac{2}{3}} + O(\lambda)$ in Lemma 2.3 we get

$$
\begin{aligned}
&\widetilde{U}(pu - (u_1 z_1 + u_2 z_2)) \qquad (3.9)\\
&= \frac{1-p}{p}(p\gamma_0)^{-\frac{p}{1-p}} + p\gamma_1(p\gamma_0)^{-\frac{1}{1-p}}\lambda^{\frac{1}{3}} + p\gamma_2(p\gamma_0)^{-\frac{1}{1-p}}\lambda^{\frac{2}{3}} + O\left(b^2\right).
\end{aligned}
$$

Since inside the NT region, (2.31) holds with equality, we can equate the $O(1)$ and $O\left(\lambda^{\frac{1}{3}}\right)$ terms in (3.7) and (3.9), to get

$$\gamma_0 = \frac{A^{p-1}}{p}, \quad \gamma_1 = 0. \qquad (3.10)$$

As expected γ_0 is the value function with for the case of zero transaction cost.

Substituting the values of γ_0 and γ_1 into (3.9) we find

$$\widetilde{U}(pu - (u_1 z_1 + u_2 z_2)) = \frac{1-p}{p}A^p + p\gamma_2 A\lambda^{\frac{2}{3}} + O(\lambda^{\frac{4}{3}}) \qquad (3.11)$$

3.1 Solution in Corners

3.1.1 The southwestern corner

According to Assumption 3.1 (ii), u satisfies both (2.32) and (2.33) with equality in all $\overline{SW}$ region. In particular, in SW we have the equations

$$\lambda_1 pu - (1+\lambda_1 z_1)u_1 - \lambda_1 z_2 u_2 = 0, \tag{3.12}$$
$$\lambda_2 pu - \lambda_2 z_1 u_1 - (1+\lambda_2 z_2)u_2 = 0. \tag{3.13}$$

It follows that for $(z_1, z_2) \in \overline{SW}$

$$u(z_1, z_2) = \left(\frac{1+\lambda_1 z_1 + \lambda_2 z_2}{1+\lambda_1 z_{1,SW} + \lambda_2 z_{2,SW}}\right)^p u(z_{1,SW}, z_{2,SW}), \tag{3.14}$$

where $(z_{1,SW}, z_{2,SW})$ are the coordinates of the SW corner. Then

$$\begin{aligned} u_i(z_1, z_2) &= \lambda_i p \frac{(1+\lambda z_1 + \lambda z_2)^{p-1}}{(1+\lambda_1 z_{1,SW} + \lambda_2 z_{2,SW})^p} u(z_{1,SW}, z_{2,SW}) \\ &= \frac{\lambda_i p}{1+\lambda_1 z_1 + \lambda_2 z_2} u(z_1, z_2), \quad i = 1, 2, \end{aligned} \tag{3.15}$$

$$\begin{aligned} u_{ij}(z_1, z_2) &= \lambda_i \lambda_j p(p-1) \frac{(1+\lambda_1 z_1 + \lambda_2 z_2)^{p-2}}{(1+\lambda_1 z_{1,SW} + \lambda_2 z_{2,SW})^p} u(z_{1,SW}, z_{2,SW}) \\ &= \frac{\lambda_i \lambda_j p(p-1)}{(1+\lambda_1 z_1 + \lambda_2 z_2)^2} u(z_1, z_2), \quad i, j = 1, 2. \end{aligned} \tag{3.16}$$

And in particular from Assumption 3.1 (i) it follows

$$u_i(z_{1,SW}, z_{2,SW}) = O(\lambda), \quad i = 1, 2, \tag{3.17}$$
$$u_{ij}(z_{1,SW}, z_{2,SW}) = O\left(\lambda^2\right), \quad i, j = 1, 2. \tag{3.18}$$

3.1.2 The other corners

Similarly to (3.14) we calculate

$$u(z_1, z_2) = u(z_{1,NE}, z_{2,NE}) \left(\frac{1-\lambda_1 z_1 - \lambda_2 z_2}{1-\lambda_1 z_{1,NE} - \lambda_2 z_{2,NE}}\right)^p \quad (z_1, z_2) \in \overline{NE} \tag{3.19}$$

$$u(z_1, z_2) = u(z_{1,NW}, z_{2,NW}) \left(\frac{1-\lambda_2 z_2 + \lambda_1 z_1}{1-\lambda_2 z_{2,NW} + \lambda_1 z_{1,NW}}\right)^p \quad (z_1, z_2) \in \overline{NW} \tag{3.20}$$

$$u(z_1, z_2) = u(z_{1,SE}, z_{2,SE}) \left(\frac{1-\lambda_1 z_1 + \lambda_2 z_2}{1-\lambda_1 z_{1,SE} + \lambda_2 z_{2,SE}}\right)^p \quad (z_1, z_2) \in \overline{SE} \tag{3.21}$$

Remark 3.2. Similarly to (3.17), (3.18) it follows that in all four corners, the first partial derivatives are of order λ and the second partial derivatives are of order λ^2. □

Remark 3.2 tells us that if we let $(z_1, z_2) = (z_{1,NE}, z_{2,NE})$ be the northeastern corner, then equation (2.27) becomes

$$\begin{aligned}&\mathcal{D}u(z_{1,NE}, z_{2,NE}) \qquad (3.22)\\ &= (1-p)A\gamma_0 + \frac{1-p}{2}\left[\sigma_1^2(z_{1,NE}-\theta_1)^2 + \sigma_2^2(z_{2,NE}-\theta_2)^2\right]A^{p-1} - (1-p)A\gamma_2\lambda^{\frac{2}{3}} + O(\lambda).\end{aligned}$$

Evaluation of (3.11) at $(z_{1,NE}, z_{2,NE})$ leads to

$$\widetilde{\mathcal{U}}(u)(z_{1,NE}, z_{2,NE}) = \frac{1-p}{p}A^p + p\gamma_2 A\lambda^{\frac{2}{3}} + O(\lambda). \qquad (3.23)$$

Since $(z_{1,NE}, z_{2,NE}) \in \overline{NT}$, (3.22) and (3.23) are equal. Moreover, applying (3.10) and repeating this calculation for the other three corners produces the system of equations

$$\begin{cases} \sigma_1^2(z_{1,NE}-\theta_1)^2 + \sigma_2^2(z_{2,NE}-\theta_2)^2 &= \frac{2}{1-p}A^{2-p}\gamma_2\lambda^{\frac{2}{3}} + O(\lambda),\\ \sigma_1^2(z_{1,NW}-\theta_1)^2 + \sigma_2^2(z_{2,NW}-\theta_2)^2 &= \frac{2}{1-p}A^{2-p}\gamma_2\lambda^{\frac{2}{3}} + O(\lambda),\\ \sigma_1^2(z_{1,SW}-\theta_1)^2 + \sigma_2^2(z_{2,SW}-\theta_2)^2 &= \frac{2}{1-p}A^{2-p}\gamma_2\lambda^{\frac{2}{3}} + O(\lambda),\\ \sigma_1^2(z_{1,SE}-\theta_1)^2 + \sigma_2^2(z_{2,SE}-\theta_2)^2 &= \frac{2}{1-p}A^{2-p}\gamma_2\lambda^{\frac{2}{3}} + O(\lambda). \end{cases} \qquad (3.24)$$

Based on Assumption 3.1 (v), we write

$$z_{1,NE} = z_{1,SE} + O(\lambda^{\frac{2}{3}}),\ \ z_{1,NW} = z_{1,SW} + O(\lambda^{\frac{2}{3}}),$$

$$z_{2,NE} = z_{2,NW} + O(\lambda^{\frac{2}{3}}),\ \ z_{2,SE} = z_{2,SW} + O(\lambda^{\frac{2}{3}}).$$

Let

$$c_1 = \sigma_1(z_{1,NE}-\theta_1) = \sigma_1(z_{1,SE}-\theta_1) + O(\lambda^{\frac{2}{3}}),\ \ \ c_2 = \sigma_1(z_{1,NW}-\theta_1) = \sigma_1(z_{1,SW}-\theta_1) + O(\lambda^{\frac{2}{3}});$$

$$c_3 = \sigma_2(z_{2,NE}-\theta_2) = \sigma_2(z_{2,NW}-\theta_2) + O(\lambda^{\frac{2}{3}}),\ \ \ c_4 = \sigma_2(z_{2,SE}-\theta_2) = \sigma_2(z_{2,SW}-\theta_2) + O(\lambda^{\frac{2}{3}}).$$

Then (3.24) becomes

$$\begin{cases} c_1^2 + c_3^2 &= \frac{2}{1-p}A^{2-p}\gamma_2\lambda^{\frac{2}{3}} + O(\lambda),\\ c_2^2 + c_3^2 &= \frac{2}{1-p}A^{2-p}\gamma_2\lambda^{\frac{2}{3}} + O(\lambda),\\ c_2^2 + c_4^2 &= \frac{2}{1-p}A^{2-p}\gamma_2\lambda^{\frac{2}{3}} + O(\lambda),\\ c_1^2 + c_4^2 &= \frac{2}{1-p}A^{2-p}\gamma_2\lambda^{\frac{2}{3}} + O(\lambda). \end{cases}$$

Subtracting, we get

$$\begin{cases} (c_1-c_2)(c_1+c_2) &= O(\lambda),\\ (c_3-c_4)(c_3+c_4) &= O(\lambda). \end{cases} \qquad (3.25)$$

By Assumption 3.1 (iii) we can write $c_i = a_i\lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}})$. Next we rule out the case when

$a_1 = a_2$ and the case when $a_3 = a_4$, as they both would violate Assumption 3.1 (iii), causing one of the sides of the NT region to become order $O(\lambda^{\frac{2}{3}})$. The only other case satisfying (3.25) is $c_1 + c_2 = O(\lambda^{\frac{2}{3}})$, i.e., $c_1 = -c_2 + O(\lambda^{\frac{2}{3}})$, and similarly $c_3 = -c_4 + O(\lambda^{\frac{2}{3}})$. It follows that

$$\begin{aligned} z_{1,NE} - \theta_1 &= \frac{1}{2}\nu_1\lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}}), & z_{1,SE} - \theta_1 &= \frac{1}{2}\nu_1\lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}}), \\ z_{1,NW} - \theta_1 &= -\frac{1}{2}\nu_1\lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}}), & z_{1,SW} - \theta_1 &= -\frac{1}{2}\nu_1\lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}}), \\ z_{2,NE} - \theta_2 &= \frac{1}{2}\nu_2\lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}}), & z_{2,NW} - \theta_2 &= \frac{1}{2}\nu_2\lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}}), \\ z_{2,SE} - \theta_2 &= -\frac{1}{2}\nu_2\lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}}), & z_{2,SW} - \theta_2 &= -\frac{1}{2}\nu_2\lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}}). \end{aligned} \tag{3.26}$$

By Assumption 3.1 (v), the rectangle is non-degenerate , so $\nu_1, \nu_2 > 0$.

3.1.3 Second Derivatives in Corners

From Remark 3.2 it follows that $u_{ij}(z_1, z_2) = O(\lambda^2)$ for $i, j = 1, 2$, when $(z_1, z_2) = (z_{1,SW}, z_{2,SW})$. Evaluating (3.4) at the corner, using (3.26), we get

$$\begin{aligned} &u_{11}(z_{1,SW}, z_{2,SW}) = \\ &\quad -2\gamma_{5,1}\lambda^{\frac{2}{3}} + (3\gamma_{6,1}\nu_1 + \gamma_{6,2}\nu_2)\lambda^{\frac{2}{3}} - (3\gamma_{7,1}\nu_1^2 + \frac{3}{2}\gamma_{7,2}\nu_1\nu_2 + \frac{1}{2}\gamma_{7,3}\nu_2^2)\lambda^{\frac{2}{3}} + O(\lambda) \\ &= O(\lambda^2) \end{aligned} \tag{3.27}$$

Repeating this calculation for the other three corners, we get

$$\begin{cases} -2\gamma_{5,1} - (3\gamma_{6,1}\nu_1 + \gamma_{6,2}\nu_2) - (3\gamma_{7,1}\nu_1^2 + \frac{3}{2}\gamma_{7,2}\nu_1\nu_2 + \frac{1}{2}\gamma_{7,3}\nu_2^2) &= 0, \\ -2\gamma_{5,1} - (3\gamma_{6,1}\nu_1 - \gamma_{6,2}\nu_2) - (3\gamma_{7,1}\nu_1^2 - \frac{3}{2}\gamma_{7,2}\nu_1\nu_2 + \frac{1}{2}\gamma_{7,3}\nu_2^2) &= 0, \\ -2\gamma_{5,1} + (3\gamma_{6,1}\nu_1 + \gamma_{6,2}\nu_2) - (3\gamma_{7,1}\nu_1^2 + \frac{3}{2}\gamma_{7,2}\nu_1\nu_2 + \frac{1}{2}\gamma_{7,3}\nu_2^2) &= 0, \\ -2\gamma_{5,1} + (3\gamma_{6,1}\nu_1 - \gamma_{6,2}\nu_2) - (3\gamma_{7,1}\nu_1^2 - \frac{3}{2}\gamma_{7,2}\nu_1\nu_2 + \frac{1}{2}\gamma_{7,3}\nu_2^2) &= 0. \end{cases} \tag{3.28}$$

Similarly

$$\begin{aligned} &u_{22}(z_{1,SW}, z_{2,SW}) = \\ &\quad -2\gamma_{5,3}\lambda^{\frac{2}{3}} + (3\gamma_{6,4}\nu_2 + \gamma_{6,3}\nu_1)\lambda^{\frac{2}{3}} - (3\gamma_{7,5}\nu_2^2 + \frac{3}{2}\gamma_{7,4}\nu_2\nu_1 + \frac{1}{2}\gamma_{7,3}\nu_1^2)\lambda^{\frac{2}{3}} + O(\lambda) \\ &= O(\lambda^2), \end{aligned} \tag{3.29}$$

$$\begin{aligned} &u_{12}(z_{1,SW}, z_{2,SW}) = \\ &\quad -\gamma_{5,2}\lambda^{\frac{2}{3}} + (\gamma_{6,2}\nu_1 + \gamma_{6,3}\nu_2)\lambda^{\frac{2}{3}} - (\frac{3}{4}\gamma_{7,2}\nu_1^2 + \gamma_{7,3}\nu_1\nu_2 + \frac{3}{4}\gamma_{7,4}\nu_2^2)\lambda^{\frac{2}{3}} + O(\lambda) \\ &= O(\lambda^2). \end{aligned}$$

Repeating this process for the other three corners results in two more systems of four equations each

$$\begin{cases} -2\gamma_{5,3} - (3\gamma_{6,4}\nu_2 + \gamma_{6,3}\nu_1) - (3\gamma_{7,5}\nu_2^2 + \frac{3}{2}\gamma_{7,4}\nu_2\nu_1 + \frac{1}{2}\gamma_{7,3}\nu_1^2) &= 0, \\ -2\gamma_{5,3} - (3\gamma_{6,4}\nu_2 - \gamma_{6,3}\nu_1) - (3\gamma_{7,5}\nu_2^2 - \frac{3}{2}\gamma_{7,4}\nu_2\nu_1 + \frac{1}{2}\gamma_{7,3}\nu_1^2) &= 0, \\ -2\gamma_{5,3} + (3\gamma_{6,4}\nu_2 + \gamma_{6,3}\nu_1) - (3\gamma_{7,5}\nu_2^2 + \frac{3}{2}\gamma_{7,4}\nu_2\nu_1 + \frac{1}{2}\gamma_{7,3}\nu_1^2) &= 0, \\ -2\gamma_{5,3} + (3\gamma_{6,4}\nu_2 - \gamma_{6,3}\nu_1) - (3\gamma_{7,5}\nu_2^2 - \frac{3}{2}\gamma_{7,4}\nu_2\nu_1 + \frac{1}{2}\gamma_{7,3}\nu_1^2) &= 0. \end{cases} \tag{3.30}$$

and

$$\begin{cases} -\gamma_{5,2} - (\gamma_{6,2}\nu_1 + \gamma_{6,3}\nu_2) - (\frac{3}{4}\gamma_{7,2}\nu_1^2 + \gamma_{7,3}\nu_1\nu_2 + \frac{3}{4}\gamma_{7,4}\nu_2^2) &= 0, \\ -\gamma_{5,2} - (\gamma_{6,2}\nu_1 - \gamma_{6,3}\nu_2) - (\frac{3}{4}\gamma_{7,2}\nu_1^2 - \gamma_{7,3}\nu_1\nu_2 + \frac{3}{4}\gamma_{7,4}\nu_2^2) &= 0, \\ -\gamma_{5,2} + (\gamma_{6,2}\nu_1 + \gamma_{6,3}\nu_2) - (\frac{3}{4}\gamma_{7,2}\nu_1^2 + \gamma_{7,3}\nu_1\nu_2 + \frac{3}{4}\gamma_{7,4}\nu_2^2) &= 0, \\ -\gamma_{5,2} + (\gamma_{6,2}\nu_1 - \gamma_{6,3}\nu_2) - (\frac{3}{4}\gamma_{7,2}\nu_1^2 - \gamma_{7,3}\nu_1\nu_2 + \frac{3}{4}\gamma_{7,4}\nu_2^2) &= 0. \end{cases} \tag{3.31}$$

Going back to system (3.28), we subtract the first equation from the third to get $3\gamma_{6,1}\nu_1 + \gamma_{6,2}\nu_2 = 0$, so $\gamma_{6,2} = -3\gamma_{6,1}\frac{\nu_1}{\nu_2}$. Subtracting the second equation from the first, we get $(2\gamma_{6,2} + 3\gamma_{7,2}\nu_1)\nu_2 = 0$, thus $\gamma_{6,2} = -\frac{3}{2}\gamma_{7,2}\nu_1$. Subtracting the first equation from the fourth, in the same system, we get $(6\gamma_{6,1} + 3\gamma_{7,2}\nu_2)\nu_1 = 0$, then $\gamma_{7,2} = -2\frac{\gamma_{6,1}}{\nu_2}$. Combining these results we get $-3\gamma_{6,1}\frac{\nu_1}{\nu_2} = 3\gamma_{6,1}\frac{\nu_1}{\nu_2}$, which implies

$$\gamma_{6,1} = 0 = \gamma_{6,2} = \gamma_{7,2}. \tag{3.32}$$

By repeating this process with the other two systems of equations (3.30) and (3.31), we conclude that

$$\gamma_{5,2} = 0, \tag{3.33}$$
$$\gamma_{6,1} = \gamma_{6,2} = \gamma_{6,3} = \gamma_{6,4} = 0, \tag{3.34}$$
$$\gamma_{7,2} = \gamma_{7,3} = \gamma_{7,4} = 0 \tag{3.35}$$

Notice, that the system (3.31) becomes now a zero identity and the systems of equations (3.28) and (3.30) can be summarized as

$$\begin{cases} \gamma_{7,1} &= -\frac{2}{3\nu_1^2}\gamma_{5,1}, \\ \gamma_{7,5} &= -\frac{2}{3\nu_2^2}\gamma_{5,3}. \end{cases} \tag{3.36}$$

We rewrite equations (3.1)–(3.6) as

$$\begin{aligned} u(z_1,z_2) &= \frac{A^{p-1}}{p} - \gamma_2\lambda^{\frac{2}{3}} - \gamma_3\lambda - (\gamma_{4,1}(z_1-\theta_1) + \gamma_{4,2}(z_2-\theta_2))\lambda && (3.37) \\ &\quad -(\gamma_{5,1}(z_1-\theta_1)^2 + \gamma_{5,3}(z_2-\theta_2)^2)\lambda^{\frac{2}{3}} - (\gamma_{7,1}(z_1-\theta_1)^4 + \gamma_{7,5}(z_2-\theta_2)^4) + O(\lambda^{\frac{5}{3}}), \\ u_1(z_1,z_2) &= -\gamma_{4,1}\lambda - 2\gamma_{5,1}(z_1-\theta_1)\lambda^{\frac{2}{3}} - 4\gamma_{7,1}(z_1-\theta_1)^3 + O(\lambda^{\frac{4}{3}}), && (3.38) \\ u_2(z_1,z_2) &= -\gamma_{4,2}\lambda - 2\gamma_{5,3}(z_2-\theta_2)\lambda^{\frac{2}{3}} - 4\gamma_{7,5}(z_2-\theta_2)^3 + O(\lambda^{\frac{4}{3}}), && (3.39) \\ u_{11}(z_1,z_2) &= -2\gamma_{5,1}\lambda^{\frac{2}{3}} - 12\gamma_{7,1}(z_1-\theta_1)^2 + O(\lambda), && (3.40) \\ u_{22}(z_1,z_2) &= -2\gamma_{5,3}\lambda^{\frac{2}{3}} - 12\gamma_{7,5}(z_2-\theta_2)^2 + O(\lambda), && (3.41) \\ u_{12}(z_1,z_2) &= O(\lambda). && (3.42) \end{aligned}$$

3.1.4 First Derivatives in Corners

From (3.15) it follows that

$$u_1(z_{1,SW}, z_{1,SW}) = \frac{\lambda_1 p}{1+\lambda_1 z_{1,SW}+\lambda_2 z_{2,SW}} u(z_{1,SW}, z_{1,SW}) = \lambda_1 A^{p-1} + O(\lambda^{\frac{5}{3}}),$$
$$u_2(z_{1,SW}, z_{1,SW}) = \frac{\lambda_2 p}{1+\lambda_1 z_{1,SW}+\lambda_2 z_{2,SW}} u(z_{1,SW}, z_{1,SW}) = \lambda_2 A^{p-1} + O(\lambda^{\frac{5}{3}}),$$

where the the right hand side is the result of evaluation of the expansion (3.37) of the value function u at $(z_1, z_2) = (z_{1,SW}, z_{1,SW})$. Using the expansions (3.38) and (3.39) at $(z_1, z_2) = (z_{1,SW}, z_{1,SW})$ and the fact that $\lambda_i = \alpha_i \lambda$, we compute the left hand sides and apply (3.33), (3.34) and (3.35) to get

$$\begin{cases} 8\alpha_1 A^{p-1} &= -8\gamma_{4,1} + 8\gamma_{5,1}\nu_1 + 4\gamma_{7,1}\nu_1^3, \\ 8\alpha_2 A^{p-1} &= -8\gamma_{4,2} + 8\gamma_{5,3}\nu_2 + 4\gamma_{7,5}\nu_2^3. \end{cases} \tag{3.43}$$

Doing the same in the northeastern corner results in two equations

$$\begin{cases} -8\alpha_1 A^{p-1} &= -8\gamma_{4,1} - 8\gamma_{5,1}\nu_1 - 4\gamma_{7,1}\nu_1^3, \\ -8\alpha_2 A^{p-1} &= -8\gamma_{4,2} - 8\gamma_{5,3}\nu_2 - 4\gamma_{7,5}\nu_2^3. \end{cases}$$

Combining this with (3.43), we see that $\gamma_{4,1} = \gamma_{4,2} = 0$, and the resulting two equations are

$$\begin{cases} 8\alpha_1 A^{p-1} &= 8\gamma_{5,1}\nu_1 + 4\gamma_{7,1}\nu_1^3, \\ 8\alpha_2 A^{p-1} &= 8\gamma_{5,3}\nu_2 + 4\gamma_{7,5}\nu_2^3. \end{cases} \tag{3.44}$$

The reader can verify that repeating the process for the other two corners, the northwestern and the southeastern corners, leads to the same two equations (3.44).

Combining (3.44) with (3.36), we get

$$\begin{cases} \gamma_{5,1} &= \frac{3\alpha_1 A^{p-1}}{2\nu_1}, \\ \gamma_{5,3} &= \frac{3\alpha_2 A^{p-1}}{2\nu_2}. \end{cases} \tag{3.45}$$

3.1.5 More Equations

By Assumption 3.1 (iii), $(\theta_1, \theta_2) \in \overline{NT}$, so equation (2.31) holds with equality there. Using the power expansion of the value function and its derivatives (3.37)–(3.42) and (3.11) and expanding the equality up to order $O(\lambda)$, we get by comparing the coefficients of $O\left(\lambda^{\frac{2}{3}}\right)$ that

$$A\gamma_2 = \left[\sigma_1^2\theta_1^2 + \sigma_2^2\theta_2^2\right]\theta_1^2\gamma_{5,1} + \left[\sigma_2^2\theta_2^2 + \sigma_1^2\theta_1^2\right]\theta_2^2\gamma_{5,3}. \tag{3.46}$$

Note that both of the first derivatives of u are of order $O(\lambda)$, so they are not present in (3.46). By Assumption 3.1 (v) $(z_{1,NE}, \theta_2), (\theta_1, z_{2,NE}) \in \overline{NT}$, so equation (2.31) holds with equality there. Expanding it again at these two points and using the fact that $z_{i,NE} = \theta_i + \frac{1}{2}\nu_i\lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}}),\ i = 1, 2$, we have by comparing the coefficients of order $O\left(\lambda^{\frac{2}{3}}\right)$

$$A\gamma_2 = \frac{1}{8}(1-p)\sigma_1^2\nu_1^2A^{p-1} + \left[\sigma_1^2\theta_1^2 + \sigma_2^2\theta_2^2\right]\theta_1^2\left[\gamma_{5,1} + \frac{3\gamma_{7,1}}{2}\nu_1^2\right] + \left[\sigma_2^2\theta_2^2 + \sigma_1^2\theta_1^2\right]\theta_2^2\gamma_{5,3}, \quad (3.47)$$

$$A\gamma_2 = \frac{1}{8}(1-p)\sigma_2^2\nu_2^2A^{p-1} + \left[\sigma_1^2\theta_1^2 + \sigma_2^2\theta_2^2\right]\theta_1^2\gamma_{5,1} + \left[\sigma_2^2\theta_2^2 + \sigma_1^2\theta_1^2\right]\theta_2^2\left[\gamma_{5,3} + \frac{3\gamma_{7,5}}{2}\nu_2^2\right]. \quad (3.48)$$

From (3.36), we also have

$$\gamma_{5,3} + \frac{6\gamma_{7,5}}{4}\nu_2^2 = 0 = \gamma_{5,1} + \frac{6\gamma_{7,1}}{4}\nu_1^2. \quad (3.49)$$

Combining that with (3.46), (3.47) and (3.48), we get

$$\begin{cases} \frac{1}{8}(1-p)\sigma_2^2\nu_2^2A^{p-1} &= \left[\sigma_2^2\theta_2^2 + \sigma_1^2\theta_1^2\right]\theta_2^2\gamma_{5,3}, \\ \frac{1}{8}(1-p)\sigma_1^2\nu_1^2A^{p-1} &= \left[\sigma_1^2\theta_1^2 + \sigma_2^2\theta_2^2\right]\theta_1^2\gamma_{5,1}, \\ \frac{2}{1-p}A^{2-p}\gamma_2 &= \frac{1}{4}\sigma_1^2\nu_1^2 + \frac{1}{4}\sigma_2^2\nu_2^2, \end{cases} \quad (3.50)$$

where the last equation is (3.24). We can solve (3.50) using (3.45) to get

$$\nu_i = \sqrt[3]{12\alpha_i\theta_i^2\frac{\sigma_1^2\theta_1^2 + \sigma_2^2\theta_2^2}{(1-p)\sigma_i^2}}, \quad (3.51)$$

$$\gamma_2 = A^{p-2}\sum_{i=1}^{2}\sqrt[3]{\frac{9}{32}\alpha_i^2(1-p)\sigma_i^2\theta_i^4\left(\sigma_1^2\theta_1^2 + \sigma_2^2\theta_2^2\right)^2}, \quad (3.52)$$

$$\begin{aligned} \gamma_{5,1} &= \frac{3\alpha_1}{2\nu_1}A^{p-1}, \quad \gamma_{5,3} = \frac{3\alpha_2}{2\nu_2}A^{p-1}, \\ \gamma_{7,1} &= -\frac{\alpha_1}{\nu_1^3}A^{p-1}, \quad \gamma_{7,5} = -\frac{\alpha_2}{\nu_2^3}A^{p-1}. \end{aligned}$$

Finally, we can write the power expansion for the value function,

$$u(z_1, z_2) = \frac{A^{p-1}}{p} - \gamma_2\ \lambda^{\frac{2}{3}} - \gamma_3\lambda - \sum_{i=1}^{2}\frac{A^{p-1}}{\nu_i}\alpha_i\left(\frac{3}{2}(z_i-\theta_i)^2\lambda^{\frac{2}{3}} - \frac{1}{\nu_i^2}(z_i-\theta_i)^4\right) + O\left(\lambda^{\frac{5}{3}}\right). \quad (3.53)$$

We have not found the value of γ_3, the coefficient of λ, but it turns not to be significant.

4 Sub- and Supersolutions for Independent Futures Contracts

4.1 Main results

In this section, after defining viscosity sub- and supersolutions, we state the Comparison Theorem 4.5. We proceed to build sub- and supersolutions to the HJB equation (2.26). The first main result is Theorem 4.1. It uses the Comparison Theorem to assert that the value function, which is a viscosity solution of (2.26) by Lemma 2.7, lies between the subsolution and the supersolution. Since the sub- and supersolutions differ by order $O(\lambda)$, we are able to estimate the value function.

We are now ready to formulate the first main theorem:

Theorem 4.1. Assume that $0 < p < 1$, $A > 0$, and $\rho = 0$. Then the value function u satisfies

$$u(\theta_1, \theta_2) = \frac{A^{p-1}}{p} - A^{p-2}\lambda^{\frac{2}{3}} \sum_{i=1}^{2} \sqrt[3]{\frac{9}{32}\alpha_i^2(1-p)\sigma_i^2\theta_i^4\left(\sigma_1^2\theta_1^2 + \sigma_2^2\theta_2^2\right)^2} + O(\lambda). \tag{4.1}$$

In this section we will concentrate on proving this theorem in the case of $\rho = 0$. The case when $\rho \neq 0$ is the subject of the next section.

Proof of the Main Theorem 4.1: This proof is divided into multiple steps. We first divide the solvency region $\mathcal{S}_u$ into nine regions, similar to Figure 1. The $NT^{\pm}$ region in Definition 4.14, is constructed in Step 1 in such a way that the viscosity sub- and supersolutions can be extended in a C^1 fashion to the rest of the solvency region. In Section 4.2 viscosity sub- and supersolutions are defined, and based on the arguments in the Appendix we explain why they are also smooth. In Sections 4.3, 4.4 and 4.5 we prove that inside each of the sub-regions comprising the solvency region, the viscosity sub- and supersolution conditions (4.3) and (4.4) respectively hold. We finish by applying Lemma 4.7 that shows that the viscosity sub- and supersolution conditions are satisfied on the boundary of these sub-regions.

Corollary 4.2. Assume $0 < p < 1$ and $A > 0$. Then for fixed $(z_1, z_2) \in \mathcal{S}_u$, the value function satisfies

$$u(z_1, z_1) = \frac{1}{p}A^{p-1} - A^{p-2} \sum_{i=1}^{2} \sqrt[3]{\frac{9}{32}\alpha_i^2(1-p)\sigma_i^2\theta_i^4\left(\sigma_1^2\theta_1^2 + \sigma_2^2\theta_2^2\right)^2}\lambda^{\frac{2}{3}} + O(\lambda). \tag{4.2}$$

Lemma 4.3. Assume $0 < p < 1$ and $A > 0$, and assume that the solvency region is divided into nine regions appearing in Table 1, then inside the NT region, the optimal consumption

$$c = \left(A + \frac{p}{1-p}\gamma_2\lambda^{\frac{2}{3}}\right) + O(\lambda).$$

4.2 Viscosity solutions

Definition 4.4. [14], [15], [16] Let $w : \mathcal{S}_u \to R$ be continuous. We say w is a *viscosity subsolution* of (2.26) if, for every $(z_1, z_2) \in \mathcal{S}_u$ and for every $\varphi \in C^2(\mathcal{S}_u)$ satisfying $\varphi \geq w$ on $\mathcal{S}_u$ and $\varphi(z_1, z_2) = w(z_1, z_2)$, we have

$$(\mathcal{H}(\varphi))(z_1, z_2) \leq 0. \tag{4.3}$$

We say w is a *viscosity supersolution* of (2.26) if, for every $(z_1, z_2) \in \mathcal{S}_u$ and for every $\varphi \in C^2(\mathcal{S}_u)$ satisfying $\varphi \leq w$ on $\mathcal{S}_u$ and $\varphi(z_1, z_2) \in \mathcal{S}_u = w(z_1, z_2) \in \mathcal{S}_u$, we have

$$(\mathcal{H}(\varphi))(z_1, z_2) \geq 0. \tag{4.4}$$

We say w is a *viscosity solution* of (2.26) if it is both a viscosity subsolution and a viscosity supersolution.

Comparison Theorem 4.5. Assume $0 < p < 1$, $\lambda_i > 0$, $i = 1, 2$ and $A > 0$. Let a continuous function $\psi^{\pm}$ be a supersolution (subsolution respectively) to (2.26) $\psi^{\pm} > 0$ in $\mathcal{S}_u$ with the boundary condition $\psi^{+}|_{\partial\mathcal{S}_u} \geq \psi^{-}|_{\partial\mathcal{S}_u} = 0$, then $\psi^{+} \geq \psi^{-}$, then $\psi^{+} \geq \psi^{-}$. In particular, any supersolution majorizes the function u defined by (2.23), and any subsolution minorizes u.

The proof is left to the Appendix.

Remark 4.6. If a viscosity solution w of (2.26) is in $C^2(\mathcal{S}_u)$, then w is a solution in the classical sense. To see this, take the test function φ in Definition 4.4 to be w itself and use (4.3) and (4.4) to obtain $\mathcal{H}(w) = 0$.

Moreover, if $w \in C^2(\mathcal{S}_u)$ is a classical solution of (2.26), then it is also a viscosity solution. To see the subsolution property, let $\varphi \in C^2(\mathcal{S}_u)$ satisfy $\varphi \geq w$ and $\varphi(z_1, z_2) = w(z_1, z_2)$. Since $\varphi - w$ attains a global minimum at (z_1, z_2), we must have $D\varphi(z_1, z_2) = Dw(z_1, z_2)$ and $D^2\varphi(z_1, z_2) \geq D^2 w(z_1, z_2)$. Thus, we have $-\frac{1}{2}(z_1, z_2)D^2\varphi(z_1, z_2)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \leq -\frac{1}{2}(z_1, z_2)D^2 w(z_1, z_2)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, so $(\mathcal{D}(\varphi))(z_1, z_2) \leq (\mathcal{D}(w))(z_1, z_2)$ and

$$(\mathcal{H}(\varphi))(z_1, z_2) \leq (\mathcal{H}(w))(z_1, z_2) = 0.$$

A similar argument establishes the supersolution property. □

Lemma 4.7. Let $\mathcal{S}_u$ be partitioned into finitely many disjoint open sets $\mathcal{O}_1, \mathcal{O}_2, ..., \mathcal{O}_n$ so that $\overline{\mathcal{S}_u} = \bigcup_{i=1}^{n} \overline{\mathcal{O}_i}$. Suppose $u \in C(\overline{\mathcal{S}_u}) \cap C^1(\mathcal{S}_u)$ is zero on $\partial\mathcal{S}_u$ and C^2 on each $\mathcal{O}_i$, and the second derivative limits

$$u_{ij}^{k}(\overline{z_1}, \overline{z_2}) \triangleq \lim_{\substack{(z_1, z_2) \in \mathcal{O}_k \\ (z_1, z_2) \to (\overline{z_1}, \overline{z_2})}} u_{ij}(z_1, z_2)$$

are defined and finite for all $(\overline{z_1}, \overline{z_2}) \in \overline{\mathcal{O}_k} \backslash \partial \mathcal{S}_u,\ k = 1, .., n.$

If

$$\min \left\{ \left(\mathcal{D} - \widetilde{\mathcal{U}} \right)(u^k), \mathcal{B}_1 u^k, \mathcal{B}_2 u^k, \mathcal{S}_1 u^k, \mathcal{S}_2 u^k \right\} \leq 0 \text{ on } \overline{\mathcal{O}_k} \backslash \partial \mathcal{S}_u,\ k = 1, .., n, \tag{4.5}$$

then u is a viscosity subsolution of (2.26).

If

$$\min \left\{ \left(\mathcal{D} - \widetilde{\mathcal{U}} \right)(u^k), \mathcal{B}_1 u^k, \mathcal{B}_2 u^k, \mathcal{S}_1 u^k, \mathcal{S}_2 u^k \right\} \geq 0 \text{ on } \overline{\mathcal{O}_k} \backslash \partial \mathcal{S}_u,\ k = 1, .., n, \tag{4.6}$$

then u is a viscosity supersolution of (2.26).

PROOF: Assume first that inequality (4.6) holds. We will prove that u is a supersolution.

Let $(z_1, z_2) \in \mathcal{S}_u$, let $\varphi \in C^2(\mathcal{S}_u)$ satisfy $\varphi \leq u$ and $\varphi(z_1, z_2) = u(z_1, z_2)$.

Case 1: $(z_1, z_2) \in \mathcal{S}_u \cap \mathcal{O}_k$ is (strictly inside) the $\mathcal{O}_k$ region. Since $\varphi \in C^2(\mathcal{S}_u)$ and $\varphi - u$ attains a global maximum at (z_1, z_2), we must have $\nabla \varphi(z_1, z_2) = \nabla u(z_1, z_2)$ and $\nabla^2 \varphi(z_1, z_2) \leq \nabla^2 u(z_1, z_2)$. We conclude that $\mathcal{S}_i(\varphi)(z_1, z_2) = \mathcal{S}_i(u)(z_1, z_2)$, and $\mathcal{B}_i(\varphi)(z_1, z_2) = \mathcal{B}_i(u)(z_1, z_2)$, for $i = 1, 2$, and similarly $\left(\mathcal{D} - \widetilde{\mathcal{U}} \right)(\varphi)(z_1, z_2) \geq \left(\mathcal{D} - \widetilde{\mathcal{U}} \right)(u)(z_1, z_2)$.

It then follows that the supersolution property (4.4) holds.

Case 2: (z_1, z_2) is on one of boundaries of the $\mathcal{O}_k$ and $\mathcal{O}_j$ regions.

Note that if (z_1, z_2) is on the boundary of only one region, that means that it is also on the boundary of the solvency region. In this case, there is nothing to verify in the supersolution property (4.4).

Since $u \in C^1(\mathcal{S}_u)$, we still have as before

$$\left(\mathcal{S}_i(\varphi) \right)(z_1, z_2), \left(\mathcal{B}_i(\varphi) \right)(z_1, z_2) \geq 0,\ i = 1, 2.$$

To conclude that u is a supersolution, we need to show that $\left(\mathcal{D} - \widetilde{\mathcal{U}} \right)(\varphi)(z_1, z_2) \geq 0$. We have that $-\frac{1}{2}(z_1, z_2)\nabla^2 \varphi(z_1, z_2) \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ must be greater than or equal to the minimum of

$$\lim_{\substack{(\overline{z_1}, \overline{z_2}) \in \mathcal{O}_k \\ (\overline{z_1}, \overline{z_2}) \to (z_1, z_2)}} -\frac{1}{2}(\overline{z_1}, \overline{z_2})\nabla^2(u^k)(\overline{z_1}, \overline{z_2}) \begin{pmatrix} \overline{z_1} \\ \overline{z_2} \end{pmatrix},$$

and

$$\lim_{\substack{(\underline{z_1}, \underline{z_2}) \in \mathcal{O}_j \\ (\underline{z_1}, \underline{z_2}) \to (z_1, z_2)}} -\frac{1}{2}(\underline{z_1}, \underline{z_2})\nabla^2(u^j)(\underline{z_1}, \underline{z_2}) \begin{pmatrix} \underline{z_1} \\ \underline{z_2} \end{pmatrix}.$$

Since by assumption (4.6) $\left(\mathcal{D}-\widetilde{\mathcal{U}}\right)(u^k)(z_1,z_2)$, $\left(\mathcal{D}-\widetilde{\mathcal{U}}\right)(u^j)(z_1,z_2) \geq 0$ is non negative, then so is $\left(\mathcal{D}-\widetilde{\mathcal{U}}\right)(\varphi)(z_1,z_2)$, and the supersolution property (4.4) holds.

The case when (z_1,z_2) is on the boundary of three or more regions is treated similarly. This completes the proof for the supersolution case. The proof for the subsolution case is similar, so it is omitted.

4.3 Construction of candidate sub- and supersolutions.

Fix $B>0$ positive constant and define

$$h_i(\delta) = \alpha_i \left(\frac{3}{2}\,\delta^2\lambda^{\frac{2}{3}} - \frac{1}{\nu_i^2}\delta^4 + \frac{3}{2}B\delta^2\lambda^{\frac{4}{3}}\right).$$

We also define functions

$$\begin{aligned}
f_N^{\pm}(\delta_1,\delta_2) &= \lambda_2 - p\gamma_2 A^{1-p}\lambda_2\lambda^{\frac{2}{3}} \pm pMA^{1-p}\lambda_2\lambda - p\lambda_2\sum_{i=1}^{2}\frac{h_i(\delta_i)}{\nu_i} \\
&\quad -(1-\lambda_2(\delta_2+\theta_2))\frac{h_2'(\delta_2)}{\nu_2} + \lambda_2(\delta_1+\theta_1)\frac{h_1'(\delta_1)}{\nu_1}, \qquad (4.7)\\
f_S^{\pm}(\delta_1,\delta_2) &= \lambda_2 - p\gamma_2 A^{1-p}\lambda_2\lambda^{\frac{2}{3}} \pm pMA^{1-p}\lambda_2\lambda - p\lambda_2\sum_{i=1}^{2}\frac{h_i(\delta_i)}{\nu_i} \\
&\quad +(1+\lambda_2(\delta_2+\theta_2))\frac{h_2'(\delta_2)}{\nu_2} + \lambda_2(\delta_1+\theta_1)\frac{h_1'(\delta_1)}{\nu_1}, \qquad (4.8)\\
f_W^{\pm}(\delta_1,\delta_2) &= \lambda_1 - p\gamma_2 A^{1-p}\lambda_1\lambda^{\frac{2}{3}} \pm pMA^{1-p}\lambda_1\lambda - p\lambda_1\sum_{i=1}^{2}\frac{h_i(\delta_i)}{\nu_i} \\
&\quad +(1+\lambda_1(\delta_1+\theta_1))\frac{h_1'(\delta_1)}{\nu_1} + \lambda_1(\delta_2+\theta_2)\frac{h_2'(\delta_2)}{\nu_2}, \qquad (4.9)\\
f_E^{\pm}(\delta_1,\delta_2) &= \lambda_1 - p\gamma_2 A^{1-p}\lambda_1\lambda^{\frac{2}{3}} \pm pMA^{1-p}\lambda_1\lambda - p\lambda_1\sum_{i=1}^{2}\frac{h_i(\delta_i)}{\nu_i} \\
&\quad -(1-\lambda_1(\delta_1+\theta_1))\frac{h_1'(\delta_1)}{\nu_1} + \lambda_1(\delta_2+\theta_2)\frac{h_2'(\delta_2)}{\nu_2}. \qquad (4.10)
\end{aligned}$$

We leave the choice of the positive constant M for later.

Lemma 4.8. There exists a C^2 function

$\delta_{S,2}^{\pm}(\delta_1) = -\frac{1}{2}\nu_2\lambda^{\frac{1}{3}}(1-\xi\lambda^{\frac{1}{3}})+O(\lambda)$, where $\xi = \sqrt{\frac{2}{3}p\gamma_2A^{1-p}+B}$, such that $f_S^{\pm}(\delta_1,\delta_{S,2}^{\pm}(\delta_1))=0$ for every $\delta_1 \in [-\frac{1}{2}\nu_1\lambda^{\frac{1}{3}}, \frac{1}{2}\nu_1\lambda^{\frac{1}{3}}]$.

Proof: If $\delta_i = O(\lambda^{\frac{1}{3}}),\ i = 1,2$, then $h_i(\delta_i) = O(\lambda^{\frac{4}{3}}),\ h_i'(\delta_i) = O(\lambda),\ h_i''(\delta_i) = O(\lambda^{\frac{2}{3}})$, and

$$\begin{aligned}\frac{\partial}{\partial \delta_2} f_S^{\pm}(\delta_1,\delta_2) &= \frac{h_2''(\delta_2)}{\nu_2} + O(\lambda^{\frac{5}{3}}) \\ &= \frac{\alpha_2}{\nu_2}\left(3\lambda^{\frac{2}{3}} - \frac{12}{\nu_2^2}\delta_2^2 + 3B\lambda^{\frac{4}{3}}\right) + O(\lambda^{\frac{5}{3}}).\end{aligned} \tag{4.11}$$

So for $\delta_2 \in [-\frac{1}{2}\nu_2\lambda^{\frac{1}{3}}, \frac{1}{2}\nu_2\lambda^{\frac{1}{3}}]$, it follows that $\frac{\partial}{\partial\delta_2} f_S^{\pm}(\delta_1,\delta_2) \geq 3\frac{\alpha_2}{\nu_2}B\lambda^{\frac{4}{3}} + O(\lambda^{\frac{5}{3}})$, which is positive, for sufficiently small $\lambda > 0$.
Fix $\delta_1 \in [-\frac{1}{2}\nu_1\lambda^{\frac{1}{3}}, \frac{1}{2}\nu_1\lambda^{\frac{1}{3}}]$. For $|\delta_2| \leq \frac{1}{2}\nu_2\lambda^{\frac{1}{3}}$, we have:

$$\begin{aligned}f_S^{\pm}(\delta_1,\delta_2) &= \lambda_2 - p\gamma_2 A^{1-p}\lambda_2\lambda^{\frac{2}{3}} + \frac{h_2'(\delta_2)}{\nu_2} + O(\lambda^2) \\ &= \lambda_2 - p\gamma_2 A^{1-p}\lambda_2\lambda^{\frac{2}{3}} + \frac{\alpha_2\delta_2}{\nu_2}\left(3\lambda^{\frac{2}{3}} - \frac{4}{\nu_2^2}\delta_2^2 + 3B\lambda^{\frac{4}{3}}\right) + O(\lambda^2).\end{aligned}$$

Consider $\delta_0 = -\frac{1}{2}\nu_2\lambda^{\frac{1}{3}}(1-\xi_0\lambda^{\frac{1}{3}})$, with ξ_0 defined below, and use the fact that $\lambda_2 = \alpha_2\lambda$ to get

$$\begin{aligned}f_S^{\pm}(\delta_1,\delta_0) &= \lambda_2 - p\gamma_2 A^{1-p}\lambda_2\lambda^{\frac{2}{3}} - \frac{1}{2}\alpha_2\lambda^{\frac{1}{3}}(1-\xi_0\lambda^{\frac{1}{3}})\left(3\lambda^{\frac{2}{3}} - \lambda^{\frac{2}{3}}(1-\xi_0\lambda^{\frac{1}{3}})^2 + 3B\lambda^{\frac{4}{3}}\right) + O(\lambda^2) \\ &= \alpha_2\left(\frac{3}{2}\xi_0^2 - p\gamma_2 A^{1-p} - \frac{3}{2}B\right)\lambda^{\frac{5}{3}} + O(\lambda^2).\end{aligned} \tag{4.12}$$

With $\xi = \sqrt{\frac{2}{3}p\gamma_2 A^{1-p} + B} > 0$, we take $\xi_0 = \sqrt{\xi^2+\eta}$, where $|\eta| < \xi^2$, then

$$f_S^{\pm}(\delta_1,\delta_0) = \frac{3}{2}\alpha_2\eta\lambda^{\frac{5}{3}} + O(\lambda^2). \tag{4.13}$$

Thus, for $\eta > 0$ we have $f_S^{\pm}(\delta_1,\delta_0) > 0$ and for $\eta < 0$ we have $f_S^{\pm}(\delta_1,\delta_0) < 0$, all for sufficiently small λ. Thus $\exists\ \xi_S^{\pm}(\delta_1) = \xi + O\left(\lambda^{\frac{1}{3}}\right)$ such that for $\delta_0 = -\frac{1}{2}\nu_2\lambda^{\frac{1}{3}}(1-\xi_S^{\pm}(\delta_1)\lambda^{\frac{1}{3}})$, $f_S^{\pm}(\delta_1,\delta_0) = 0$. Note that the terms that depend on δ_1 in (4.12) are of order $O(\lambda^2)$, so they are not explicitly stated, and the same is true of $\xi_S^{\pm}(\delta_1)$, whose dependency on δ_1 is of order $O\left(\lambda^{\frac{1}{3}}\right)$.

Set $\delta_{S,2}^{\pm}(\delta_1) \triangleq -\frac{1}{2}\nu_2\lambda^{\frac{1}{3}}(1-\xi_S^{\pm}(\delta_1)\lambda^{\frac{1}{3}})$, so that $f_S^{\pm}(\delta_1,\delta_{S,2}^{\pm}(\delta_1)) = 0$ as desired. From (4.11) the condition of the implicit function theorem is satisfied and thus $\delta_{S,2}^{\pm} \in C^2\left(\left[-\frac{1}{2}\nu_1\lambda^{\frac{1}{3}}, \frac{1}{2}\nu_1\lambda^{\frac{1}{3}}\right]\right)$. This proves the claim. □

Remark 4.9. Analogously, it can be shown that there exist C^2 functions

$\delta_{N,2}^{\pm}(\delta_1) = \frac{1}{2}\nu_2\lambda^{\frac{1}{3}}(1-\xi\lambda^{\frac{1}{3}}) + O(\lambda)$, $\delta_{E,1}^{\pm}(\delta_2) = \frac{1}{2}\nu_1\lambda^{\frac{1}{3}}(1-\xi\lambda^{\frac{1}{3}}) + O(\lambda)$ and $\delta_{W,1}^{\pm}(\delta_2) = -\frac{1}{2}\nu_1\lambda^{\frac{1}{3}}(1-\xi\lambda^{\frac{1}{3}}) + O(\lambda)$ such that $f_N^{\pm}(\delta_1,\delta_{N,2}^{\pm}(\delta_1)) = 0$, $f_E^{\pm}(\delta_{E,1}^{\pm}(\delta_2),\delta_2) = 0$ and $f_W^{\pm}(\delta_{W,1}^{\pm}(\delta_2),\delta_2) = 0$.

Remark 4.10. For future reference we note that if for $|\delta_i| \le \frac{1}{2}\nu_i\lambda^{\frac{1}{3}},\ i = 1,2$, and given

$$g(\delta_1,\delta_2) = \lambda_2 - p\gamma_2 A^{1-p}\lambda_2\lambda^{\frac{2}{3}} \pm \frac{\alpha_2\delta_2}{\nu_2}\left(3\lambda^{\frac{2}{3}} - \frac{4}{\nu_2^2}\delta_2^2 + 3B\lambda^{\frac{4}{3}}\right) + O(\lambda^2)$$

Then for $\delta_0 = \mp\frac{1}{2}\nu_2\lambda^{\frac{1}{3}}(1-\xi_0\lambda^{\frac{1}{3}})$, where $\xi = \sqrt{\frac{2}{3}p\gamma_2 A^{1-p} + B}$, we take $\xi_0 = \sqrt{\xi^2+\eta}$, where $|\eta| < \xi^2$, we have

$$g(\delta_1,\delta_0) = \frac{3}{2}\alpha_2\eta\lambda^{\frac{5}{3}} + O(\lambda^2). \tag{4.14}$$

Corollary 4.11. There exist a corner $(\delta^{\pm}_{1,SW},\delta^{\pm}_{2,SW})$ such that

$f^{\pm}_S(\delta^{\pm}_{1,SW},\delta^{\pm}_{2,SW}) = 0 = f^{\pm}_W(\delta^{\pm}_{1,SW},\delta^{\pm}_{2,SW})$.

Proof: From Lemma 4.8 there exist $\delta^{\pm}_{S,2} \in C^2\left(\left[-\frac{1}{2}\nu_1\lambda^{\frac{1}{3}},\frac{1}{2}\nu_1\lambda^{\frac{1}{3}}\right]\right)$, such that $f^{\pm}_S(\delta_1,\delta^{\pm}_{S,2}(\delta_1)) = 0$, and similarly there exist $\delta^{\pm}_{W,1} \in C^2\left(\left[-\frac{1}{2}\nu_2\lambda^{\frac{1}{3}},\frac{1}{2}\nu_2\lambda^{\frac{1}{3}}\right]\right)$, such that $f^{\pm}_W(\delta^{\pm}_{W,1}(\delta_2),\delta_2) = 0$. Since both of these functions $\delta^{\pm}_{S,2}(\delta_1)$ and $\delta^{\pm}_{W,1}(\delta_2)$ are continuous and are approximately equal to $-\frac{1}{2}\nu_2\lambda^{\frac{1}{3}}(1-\xi\lambda^{\frac{1}{3}}) + O(\lambda)$ and $-\frac{1}{2}\nu_1\lambda^{\frac{1}{3}}(1-\xi\lambda^{\frac{1}{3}}) + O(\lambda)$ respectively, they necessarily intersect. There might be more then one intersection point, but we know that all the intersection points, up to order $O(\lambda)$, are all $(-\frac{1}{2}\nu_1\lambda^{\frac{1}{3}}(1-\xi\lambda^{\frac{1}{3}}), -\frac{1}{2}\nu_2\lambda^{\frac{1}{3}}(1-\xi\lambda^{\frac{1}{3}}))$. For our approximations we only use values up to $O(\lambda^{\frac{2}{3}})$ accuracy, we pick any point from this set. We call this intersection point $(\delta^{\pm}_{1,SW},\delta^{\pm}_{2,SW})$. Moreover, we define the southwestern corner $(z^{\pm}_{1,SW},z^{\pm}_{2,SW}) = (\theta_1,\theta_2) + (\delta^{\pm}_{1,SW},\delta^{\pm}_{2,SW})$.

□

Remark 4.12. Similarly to Lemma 4.8 and Corollary 4.11, the existence of three additional corners follows analogously.

Remark 4.13. For $|\delta_1| \le \frac{1}{2}\nu_1\lambda^{\frac{1}{3}}$ and $\delta_2 = -\frac{1}{2}\nu_2\lambda^{\frac{1}{3}}(1-\xi\lambda^{\frac{1}{3}}) + O(\lambda)$, from (4.8) we compute

$$\frac{\partial}{\partial\delta_1} f^{\pm}_S(\delta_1,\delta_2) = \lambda_2(\delta_1+\theta_1)\frac{h_1''(\delta_1)}{\nu_1} + O(\lambda^2) = O(\lambda^{\frac{5}{3}}). \tag{4.15}$$

Similarly, from (4.11) we have $\frac{\partial}{\partial\delta_2} f^{\pm}_S(\delta_1,\delta_2)\Big|_{\delta_2=\delta^{\pm}_{S,2}(\delta_1)} = O(\lambda)$. It follows that

$$\frac{d}{d\delta_1}\delta^{\pm}_{S,2}(\delta_1) = -\frac{\frac{\partial}{\partial\delta_1} f^{\pm}_S(\delta_1,\delta_2)}{\frac{\partial}{\partial\delta_2} f^{\pm}_S(\delta_1,\delta_2)}\Bigg|_{\delta_2=\delta^{\pm}_{S,2}(\delta_1)} = O(\lambda^{\frac{2}{3}}) \tag{4.16}$$

We also compute

$$\begin{aligned} &\frac{d^2}{d\delta_1^2}\delta^{\pm}_{S,2}(\delta_1) \qquad (4.17)\\ = \ & -\frac{\frac{\partial^2}{\partial\delta_1^2} f^{\pm}_S(\delta_1,\delta_2) + 2\frac{\partial^2}{\partial\delta_1\partial\delta_2} f^{\pm}_S(\delta_1,\delta_2)\frac{d}{d\delta_1}\delta^{\pm}_{S,2}(\delta_1) + \frac{\partial^2}{\partial\delta_2^2} f^{\pm}_S(\delta_1,\delta_2)\left(\frac{d}{d\delta_1}\delta^{\pm}_{S,2}(\delta_1)\right)^2}{\frac{\partial}{\partial\delta_2} f^{\pm}_S(\delta_1,\delta_2)}\\ = \ & O(\lambda^{\frac{1}{3}}). \end{aligned}$$

□

Definition 4.14. We define the $NT^{\pm}$ region for super- and subsolution, respectively, as the region with corners $(z^{\pm}_{1,SW}, z^{\pm}_{2,SW})$, $(z^{\pm}_{1,SE}, z^{\pm}_{2,SE})$, $(z^{\pm}_{1,NW}, z^{\pm}_{2,NW})$ and $(z^{\pm}_{1,NE}, z^{\pm}_{2,NE})$ and with southern boundary defined by $(\delta_1 + \theta_1, \delta^{\pm}_{S,2}(\delta_1) + \theta_2)$, for $\delta_1 \in [\delta^{\pm}_{1,SW}, \delta^{\pm}_{1,SE}]$, and the other boundaries defined similarly.

Remark 4.15. Note the $NT^{\pm}$ regions is almost rectangular, since the boundaries are almost straight lines parallel to the axes up to the order $O(\lambda)$.

Definition 4.16. We divide the solvency region $\mathcal{S}_u$ into $NT^{\pm}$, northern, southern, eastern, western, northeastern, southeastern, northwestern and southwestern regions, by connecting with straight lines each of the four corners of the $NT^{\pm}$ region with two appropriate corners of the solvency region. That is connecting the northeastern corner $(z^{\pm}_{1,NE}, z^{\pm}_{2,NE})$ with $(0, \frac{1}{\lambda})$ and $(\frac{1}{\lambda}, 0)$, northwestern corner $(z^{\pm}_{1,NW}, z^{\pm}_{2,NW})$ corner with $(0, \frac{1}{\lambda})$ and $(-\frac{1}{\lambda}, 0)$, southwestern corner $(z^{\pm}_{1,SW}, z^{\pm}_{2,SW})$ with $(0, -\frac{1}{\lambda})$ and $(-\frac{1}{\lambda}, 0)$ and southeastern corner $(z^{\pm}_{1,SE}, z^{\pm}_{2,SE})$ with $(0, -\frac{1}{\lambda})$ and $(\frac{1}{\lambda}, 0)$. This is the same way these regions were constructed in Figure 1, only now we do this construction for sub- and supersolutions, rather then for the solution of the HJB equation.

Define, for $(z_1, z_2) \in \overline{NT^{\pm}}$

$$w^{\pm}(z_1, z_2) = \frac{A^{p-1}}{p} - \gamma_2\ \lambda^{\frac{2}{3}} \pm M\lambda - \sum_{i=1}^{2} \frac{A^{p-1}}{\nu_i} h_i(z_i - \theta_i) \tag{4.18}$$

The reader can verify that if M were zero, then in the $NT^{\pm}$ region the formula for $w^{\pm}(z_1, z_2)$ agrees with the power series expansion (3.53) excluding the γ_3 term. The term $\pm M\lambda$ in the definition of $w^{\pm}$ will be used to create super- and subsolution.

Remark 4.17. In line with notation used in definitions from Appendix B, for $\widehat{z}^{\pm}_{1,S} \in [z^{\pm}_{1,SW}, z^{\pm}_{1,SE}]$ we define $\widehat{z}^{\pm}_{2,S}$ by

$$(\widehat{z}^{\pm}_{1,S}, \widehat{z}^{\pm}_{2,S}) = (\widehat{z}^{\pm}_{1,S}, \delta^{\pm}_{S,2}(\widehat{z}^{\pm}_{1,S} - \theta_1) + \theta_2). \tag{4.19}$$

We see from Definition 4.16 for $\widehat{z}^{\pm}_{1,S} \in [z^{\pm}_{1,SW}, z^{\pm}_{1,SE}]$ the point $(\widehat{z}^{\pm}_{1,S}, \widehat{z}^{\pm}_{2,S})$ is a point on the southern boundary of the $NT^{\pm}$ region. To reduce cumbersomeness, we will drop the boundary subscript, as long as it's clear which boundary is being referred to. For instance, we see that from Lemma 4.8 that Assumption B.1.(i) is satisfied, and that

$$f^{\pm}_S(\widehat{z}^{\pm}_1 - \theta_1, \widehat{z}^{\pm}_2 - \theta_2) = 0. \tag{4.20}$$

In this case, the reference in $\widehat{z}^{\pm}_i$ is to the southern boundary. Moreover, note that having $f^{\pm}_S(\widehat{z}^{\pm}_1 - \theta_1, \widehat{z}^{\pm}_2 - \theta_2) = 0$ on the southern boundary is equivalent to $(\mathcal{B}_2(w^{\pm}))\,(\widehat{z}^{\pm}_1, \widehat{z}^{\pm}_2) = 0$ there, which satisfies the additional requirement of Theorem B.2. Similar result holds on the other three boundaries.

To see that Assumption B.1.(iii) is also satisfied, we note that by Proposition B.4, specifically equation (B.12), two points (z_1, z_2) and (z_1', z_2') re on the same characteristic line if and only if

$$\frac{1+\lambda z_2}{z_1} = \frac{1+\lambda z_2'}{z_1'}.$$

We need to check that $\psi(\delta_1) \triangleq \frac{1+\lambda_2(\delta^{\pm}_{S,2}(\delta_1)+\theta_2)}{\delta_1+\theta_1}$ is strictly monotone for $\delta_1 \in [\delta^{\pm}_{1,SW}, \delta^{\pm}_{1,SW}]$, so that distinct points on the southern boundary of $NT^{\pm}$ are on different characteristic lines.

We compute

$$\begin{aligned}\psi'(\delta_1) &= \frac{\lambda_2 \frac{d}{d\delta_1}\delta^{\pm}_{S,2}(\delta_1)}{\delta_1+\theta_1} - \frac{1+\lambda_2(\delta^{\pm}_{S,2}(\delta_1)+\theta_2)}{(\delta_1+\theta_1)^2} \\ &= \frac{1}{(\delta_1+\theta_1)^2}\left[(\delta_1+\theta_1)\lambda_2\frac{d}{d\delta_1}\delta^{\pm}_{S,2}(\delta_1) - 1 - \lambda_2(\delta^{\pm}_{S,2}(\delta_1)+\theta_2)\right] \\ &= -\frac{1}{(\delta_1+\theta_1)^2} + O(\lambda).\end{aligned}$$

This is negative for λ sufficiently small.

Remark 4.17 shows that Assumption B.1 and all the requirements of Theorem B.2 are satisfied and we use it to extend $w^{\pm} \in C^2(\overline{NT^{\pm}})$ to the rest of the solvency. For the rest of this section, we will refer to $w^{\pm}$ as the extended function. From Theorem B.2 we conclude that $w^{\pm} \in C^1(\mathcal{S}_u)$ with zero boundary condition on the boundary of the solvency region.

4.4 Verification that $\pm\left(\mathcal{D} - \widetilde{\mathcal{U}}\right) w^{\pm} \geq 0$ in $\overline{NT^{\pm}}$.

We need to show that

$$\pm\left(\mathcal{D}w^{\pm}(z_1,z_2) - \widetilde{U}(pw^{\pm}(z_1,z_2) - (z_1 w_1^{\pm}(z_1,z_2) + z_2 w_2^{\pm}(z_1,z_2)))\right) \geq 0 \quad (z_1,z_2) \in \overline{NT^{\pm}}. \tag{4.21}$$

We use the facts that $z_i - \theta_i = O(\lambda^{1/3})$, so $h_i(z_i - \theta_i) = O(\lambda^{4/3})$, $h_i'(z_i - \theta_i) = O(\lambda)$. We have $pw^{\pm}(z_1,z_2) - (z_1 w_1^{\pm}(z_1,z_2) + z_2 w_2^{\pm}(z_1,z_2) = a - b$, where $a = A^{p-1}$ and $b = p\gamma_2\lambda^{2/3} \mp pM\lambda - \sum_{i=1}^{2} \frac{A^{p-1}}{\nu_i} z_i h_i'(z_i - \theta_i) + O(\lambda^{4/3})$. Lemma 2.3 implies:

$$
\begin{aligned}
&\widetilde{U}\big(pw^{\pm}(z_1,z_2)-(z_1w_1^{\pm}(z_1,z_2)+z_2w_2^{\pm}(z_1,z_2))\big) \qquad (4.22)\\
&= \frac{1-p}{p}A^p + A\left[p\gamma_2\lambda^{2/3} \mp pM\lambda - \sum_{i=1}^{2}\frac{A^{p-1}}{\nu_i}z_ih_i'(z_i-\theta_i)\right] + O\big(\lambda^{\frac{4}{3}}\big)\\
&= (1-p)Aw^{\pm}(z_1,z_2) + \gamma_2A\lambda^{\frac{2}{3}} \mp MA\lambda - \sum_{i=1}^{2}\frac{A^p}{\nu_i}z_ih_i'(z_i-\theta_i) + O\big(\lambda^{\frac{4}{3}}\big).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\mathcal{D}w^{\pm}(z_1,z_2) - \widetilde{U}\big(pw^{\pm}(z_1,z_2)-(z_1w_1^{\pm}(z_1,z_2)+z_2w_2^{\pm}(z_1,z_2))\big) \qquad (4.23)\\
&= \frac{1}{2}p(1-p)\sum_{i=1}^{2}\sigma_i^2(z_i-\theta_i)^2w^{\pm}(z_1,z_2)\\
&\quad + \left[z_1w_1^{\pm}(z_1,z_2)+z_2w_2^{\pm}(z_1,z_2)\right]\left[r+\sum_{i=1}^{2}\mu_iz_i-(1-p)(\sigma_1^2z_1^2+\sigma_2^2z_2^2)\right]\\
&\quad -\frac{1}{2}\left(\sigma_1^2z_1^2+\sigma_2^2z_2^2\right)\sum_{i=1}^{2}z_i^2w_{ii}^{\pm} - \gamma_2A\lambda^{\frac{2}{3}} \pm MA\lambda + \sum_{i=1}^{2}\frac{A^p}{\nu_i}z_ih_i'(z_i-\theta_i) + O\big(\lambda^{\frac{4}{3}}\big)\\
&= \sum_{i=1}^{2}\frac{1}{2}\sigma_i^2(1-p)A^{p-1}(z_i-\theta_i)^2 + \frac{1}{2}\left[\sigma_1^2z_1^2+\sigma_2^2z_2^2\right]\left[\sum_{i=1}^{2}z_i^2\frac{A^{p-1}}{\nu_i}h_i''(z_i-\theta_i)\right]\\
&\quad -\left[\sum_{i=1}^{2}z_i\frac{A^{p-1}}{\nu_i}h_i'(z_i-\theta_i)\right]\left[r+\sum_{i=1}^{2}\mu_iz_i-(1-p)(\sigma_1^2z_1^2+\sigma_2^2z_2^2)\right]\\
&\quad -\gamma_2A\lambda^{\frac{2}{3}} \pm MA\lambda + \sum_{i=1}^{2}\frac{A^p}{\nu_i}z_ih_i'(z_i-\theta_i) + O\big(\lambda^{\frac{4}{3}}\big).
\end{aligned}
$$

Writing $z_i = \theta_i + (z_i - \theta_i)$ we derive the relation

$$
z_i^2 = \theta_i^2 + 2\theta_i(z_i-\theta_i) + O\big(\lambda^{\frac{2}{3}}\big).
$$

Using this formula, we may write the second term on the right-hand side of (4.23) as

$$
\begin{aligned}
&\frac{1}{2}\left[\sigma_1^2z_1^2+\sigma_2^2z_2^2\right]\left[\sum_{i=1}^{2}z_i^2\frac{A^{p-1}}{\nu_i}h_i''(z_i-\theta_i)\right]\\
&= \frac{1}{2}\left[\sigma_1^2\theta_1^2+\sigma_2^2\theta_2^2\right]\left[\sum_{i=1}^{2}\theta_i^2\frac{A^{p-1}}{\nu_i}h_i''(z_i-\theta_i)\right] + O(\lambda),
\end{aligned}
$$

where the coefficient in the $O(\lambda)$ term can be chosen independently of $z_i \in [\theta_i - \frac{1}{2}\nu_i\lambda^{\frac{1}{3}}, \theta_i +$

$\frac{1}{2}\nu_i\lambda^{\frac{1}{3}}],\ i=1,2$. It follows that

$$
\begin{aligned}
&\mathcal{D}w^{\pm}(z_1,z_2)-\widetilde{U}\big(pw^{\pm}(z_1,z_2)-(z_1w_1^{\pm}(z_1,z_2)+z_2w_2^{\pm}(z_1,z_2)\big)\\
&= \sum_{i=1}^{2}\left[\frac{1}{2}\sigma_i^2(1-p)A^{p-1}-\frac{6\alpha_i}{\nu_i^3}\theta_i^2A^{p-1}\left(\sigma_1^2\theta_1^2+\sigma_2^2\theta_2^2\right)\right](z_i-\theta_i)^2\\
&\quad +A\left[-\gamma_2+A^{p-2}\left(\sigma_1^2\theta_1^2+\sigma_2^2\theta_2^2\right)\sum_{i=1}^{2}\frac{3\alpha_i}{2\nu_i}\theta_i^2\right]\lambda^{\frac{2}{3}}\pm MA\lambda+O(\lambda),
\end{aligned}
$$

where again the $O(\lambda)$ term can be chosen independently of $z_i\in[\theta_i-\frac{1}{2}\nu_i\lambda^{\frac{1}{3}},\theta_i+\frac{1}{2}\nu_i\lambda^{\frac{1}{3}}],\ i=1,2$. The definitions of ν_i and γ_2 imply that the first two terms on the right-hand side are zero. We can choose M big enough so that

$$
\left(\mathcal{D}-\widetilde{\mathcal{U}}\right)(w^+)\geq 0,\ \left(\mathcal{D}-\widetilde{\mathcal{U}}\right)(w^-)\leq 0,\ \text{on }\overline{NT^{\pm}}.
$$

4.5 Conclusion of proof that w^- is a subsolution.

We finish the proof that w^- is a viscosity subsolution by applying Lemma 4.7. We only need to verify that the assumptions of Lemma 4.7 hold.

Case 1: $(z_1,z_2)\in\mathcal{S}_u\backslash NT^-$.

Then (z_1,z_2) is in one of the other eight regions, or on their boundaries. For instance, if (z_1,z_2) in the southern region or on its boundary, then $(\mathcal{B}_2(w^-))(z_1,z_2)=0$, and assumption (4.5) holds.

Case 2: $(z_1,z_2)\in NT^-$.

In this case, in Step 3 we showed that $\left(\mathcal{D}-\widetilde{\mathcal{U}}\right)(w^-)\leq 0$, and assumption (4.5) holds again.

From Lemma 4.7 it now follows that w^- is subsolution.

4.6 Conclusion of proof that w^+ is a supersolution.

4.6.1 Step 1: $\mathcal{B}_i(w^+),\mathcal{S}_i(w^+)\geq 0$ in $\overline{NT^+}$ region, $i=1,2$.

To conclude that inside the $\overline{NT^+}$ region $\mathcal{H}(w^+)\geq 0$, we still need to show that $\mathcal{B}_i(w^+),\mathcal{S}_i(w^+)\geq 0,\ i=1,2$. Consider, for example, $\mathcal{B}_2(w^+)$. We want to show that

$$
g(z_1,z_2)=\lambda_2pw^+(z_1,z_2)-(1+\lambda_2z_2)w_2^+(z_1,z_2)-\lambda_2z_1w_1^+(z_1,z_2)\geq 0,\quad (z_1,z_2)\in\overline{NT^+}, \tag{4.24}
$$

Fix $(z_1, z_2) \in \overline{NT^+}$. We have $z_i - \theta_i = O\big(\lambda^{1/3}\big)$. Thus the derivative of g with respect to z_2 is

$$\begin{aligned} g_2(z_1, z_2) &= \lambda_2 p w_2^+(z_1, z_2) - \lambda_2 w_2^+(z_1, z_2) - (1 + \lambda_2 z_2) w_{22}^+(z_1, z_2) - \lambda_2 z_1 w_{12}^+(z_1, z_2) \\ &= -w_{22}^+(z_1, z_2) + O(\lambda^{\frac{5}{3}}) = \frac{A^{p-1}}{\nu_2} h_2''(z_2 - \theta_2) + O(\lambda^{\frac{5}{3}}). \end{aligned}$$

Using this fact, we compute

$$g_2(z_1, z_2) = \frac{12\alpha_2}{\nu_2} A^{p-1} \lambda^{\frac{2}{3}} \left[\frac{1}{4} - \left(\frac{z_2 - \theta_2}{\nu_2 \lambda^{\frac{1}{3}}} \right)^2 + O(\lambda) \right]. \tag{4.25}$$

The point $(z_1, \delta_{S,2}^+(z_1 - \theta_1) + \theta_2)$ has the same first coordinate as the original point (z_1, z_2), and $z_2 - \theta_2 \geq \delta_{S,2}^+(z_1 - \theta_1)$. We also have that $f_S^+(z_1 - \theta_1, \delta_{S,2}^+(z_1 - \theta_1)) = 0$. By construction $g(z_1, \delta_{S,2}^+(z_1 - \theta_1) + \theta_2) = 0$, and to finish out proof, it's enough if we show that $g_2(z_1, z) \geq 0$ for

$$\delta_{S,2}^+(z_1 - \theta_1) + \theta_2 \leq z \leq \delta_{N,2}^+(z_1 - \theta_1) + \theta_2.$$

This means that

$$-\frac{1}{2}\nu_2 \lambda^{\frac{1}{3}}(1 - \xi\lambda^{\frac{1}{3}}) + O(\lambda) \leq z - \theta_2 \leq \frac{1}{2}\nu_2 \lambda^{\frac{1}{3}}(1 - \xi\lambda^{\frac{1}{3}}) + O(\lambda).$$

Using this, we can bound the $g_2(z_1, z)$ away from zero because

$$g_2(z_1, z) \geq \frac{12\alpha_2}{\nu_2} A^{p-1} \lambda^{\frac{2}{3}} \left[\frac{1}{2}\xi\lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}}) \right] > 0.$$

The proof that $\mathcal{B}_1(w^+)$ and $\mathcal{S}_i(w^+) \geq 0,\ i = 1, 2$, is done in a similar way.

□

4.6.2 Step 2: $\mathcal{B}_1(w^+), \mathcal{S}_i(w^+) \geq 0,\ i = 1, 2$ in the southern region.

Let (z_1, z_2) be in a point in the southern region. We want to show that $w^+(z_1, z_2)$ satisfies $\mathcal{S}_1(w^+) \geq 0$, that is,

$$\lambda_1 p w^+(z_1, z_2) + (1 - \lambda_1 z_1) w_1^+(z_1, z_2) - \lambda_1 z_2 w_2^+(z_1, z_2) \geq 0. \tag{4.26}$$

In the southern region, $w^+(z_1, z_2)$ is defined so as to satisfy $\mathcal{B}_2(w^+) = 0$, that is,

$$\lambda_2 p w^+(z_1, z_2) - (1 + \lambda_2 z_2) w_2^+(z_1, z_2) - \lambda_2 z_1 w_1^+(z_1, z_2) = 0. \tag{4.27}$$

Using notation from Lemma 4.8 and Remark 4.17, we denote $f(\widehat{z}_{1,S}^+) = (\delta_{S,2}^+)(\widehat{z}_{1,S}^+ - \theta_1) + \theta_2$. For convenience, denote also $\widehat{z}_i = \widehat{z}_{i,S}^+,\ i = 1, 2$, since in the following argument we will concentrate on the supersolution property involving only southern boundary.

Remark 4.18. Note that we already know from subsection 4.6.1 that inequality (4.26) is true for $(z_1, z_2) = (\widehat{z}_1, \widehat{z}_2)$, because $w^+ \in C^1(\mathcal{S}_u)$, and equation (4.27) holds there by construction (see Remark 4.17) It follows that $w_1^+(\widehat{z}_1, \widehat{z}_2) + \frac{\lambda_1}{\lambda_2} w_2^+(\widehat{z}_1, \widehat{z}_2) \geq 0$.

We now show inequality (4.26) holds in the rest of the southern region. Since Corollary B.20, we can use equalities (B.39) and (B.40). Together with (B.12) we have

$$\begin{aligned}
\mathcal{S}_1(w^+) &= \left(\frac{z_1}{\widehat{z}_1}\right)^p \left[\lambda_1 p w^+(\widehat{z}_1, \widehat{z}_2) + (1 - \lambda_1 z_1)\frac{\widehat{z}_1}{z_1} w_1^+(\widehat{z}_1, \widehat{z}_2) - \lambda_1 z_2 \frac{1 + \lambda_2 \widehat{z}_2}{1 + \lambda_2 z_2} w_2^+(\widehat{z}_1, \widehat{z}_2)\right] \\
&= \left(\frac{z_1}{\widehat{z}_1}\right)^p \frac{\lambda_1}{\lambda_2} \left[\lambda_2 p w^+(\widehat{z}_1, \widehat{z}_2) + \left(\frac{\lambda_2}{\lambda_1} - \lambda_2 z_1\right)\frac{\widehat{z}_1}{z_1} w_1^+(\widehat{z}_1, \widehat{z}_2) - \lambda_2 z_2 \frac{1 + \lambda_2 \widehat{z}_2}{1 + \lambda_2 z_2} w_2^+(\widehat{z}_1, \widehat{z}_2)\right]
\end{aligned}$$

Using (4.27) to substitute for $\lambda_2 p w^+(\widehat{z}_1, \widehat{z}_2)$, we obtain

$$\begin{aligned}
\mathcal{S}_1(w^+) &= \left(\frac{z_1}{\widehat{z}_1}\right)^p \frac{\lambda_1}{\lambda_2} \Big[(1 + \lambda_2 \widehat{z}_2) w_2^+(\widehat{z}_1, \widehat{z}_2) + \lambda_2 \widehat{z}_1 w_1^+(\widehat{z}_1, \widehat{z}_2) \\
&\quad + \frac{\lambda_2}{\lambda_1}\frac{\widehat{z}_1}{z_1} w_1^+(\widehat{z}_1, \widehat{z}_2) - \lambda_2 \widehat{z}_1 w_1^+(\widehat{z}_1, \widehat{z}_2) - \lambda_2 z_2 \frac{1 + \lambda_2 \widehat{z}_2}{1 + \lambda_2 z_2} w_2^+(\widehat{z}_1, \widehat{z}_2)\Big] \\
&= \left(\frac{z_1}{\widehat{z}_1}\right)^p \frac{\lambda_1}{\lambda_2} \Big[\frac{\lambda_2}{\lambda_1}\frac{\widehat{z}_1}{z_1} w_1^+(\widehat{z}_1, \widehat{z}_2) \\
&\quad + \frac{1 + \lambda_2 \widehat{z}_2 + \lambda_2 z_2 + \lambda_2^2 z_2 \widehat{z}_2 - \lambda_2 z_2 - \lambda_2^2 z_2 \widehat{z}_2}{1 + \lambda_2 z_2} w_2^+(\widehat{z}_1, \widehat{z}_2)\Big] \\
&= \left(\frac{z_1}{\widehat{z}_1}\right)^{p-1} \left[\frac{\lambda_1}{\lambda_2} w_2^+(\widehat{z}_1, \widehat{z}_2) + w_2^+(\widehat{z}_1, \widehat{z}_2)\right] \geq 0,
\end{aligned}$$

where the last inequality follows from Remark 4.18.

The proof that $\mathcal{B}_1(w^+), \mathcal{S}_2(w^+) \geq 0$, is done in a similar way.

4.6.3 Step 3: Verification that $\left(\mathcal{D} - \widetilde{\mathcal{U}}\right) w^+ \geq 0$ in the southern region.

We divide this proof into two cases.

Case 1: (z_1, z_2) is a point on the southern boundary of the NT^+ region.

This means that $(z_1, z_2) = (\widehat{z}_1, \widehat{z}_2)$. By Theorem B.2 $w^+ \in C^1(\mathcal{S}_u)$ is continuously differentiable, but possibly not twice differentiable, so for $i, j = 1, 2$ we denote

$$w_{ij}^{S,+}(\widehat{z}_1, \widehat{z}_2) \triangleq \lim_{\substack{(z_1, z_2) \in S \\ (z_1, z_2) \to (\widehat{z}_1, \widehat{z}_2)}} w_{ij}^+(z_1, z_2), \text{ and similarly}$$

$$w_{ij}^{NT,+}(\widehat{z}_1,\widehat{z}_2) \triangleq \lim_{\substack{(z_1,z_2)\in NT^+ \\ (z_1,z_2)\to(\widehat{z}_1,\widehat{z}_2)}} w_{ij}^{+}(z_1,z_2).$$

the derivatives taken from inside the southern and NT^+ regions respectively. Notice that $w_{12}^{NT,+} = 0$ and that $w_{22}^{NT,+}(\widehat{z}_1,\widehat{z}_2) = -\frac{A^{p-1}}{\nu_2}h_2''(\widehat{z}_2-\theta_2) = O(\lambda)$, since $\widehat{z}_2-\theta_2 = -\frac{1}{2}\nu_2\lambda^{\frac{1}{3}} + O\left(\lambda^{\frac{2}{3}}\right)$. Equation (4.16) allows us to use Remark B.23, from which it follows that

$$\begin{aligned} &\widehat{z}_1^2 w_{11}^{S,+}(\widehat{z}_1,\widehat{z}_2) + 2\widehat{z}_1\widehat{z}_2 w_{12}^{S,+}(\widehat{z}_1,\widehat{z}_2) + \widehat{z}_2^2 w_{22}^{S,+}(\widehat{z}_1,\widehat{z}_2) = \widehat{z}_1^2 w_{11}^{NT,+}(\widehat{z}_1,\widehat{z}_2) + O(\lambda) \\ &= \widehat{z}_1^2 w_{11}^{NT,+}(\widehat{z}_1,\widehat{z}_2) + 2\widehat{z}_1\widehat{z}_2 w_{12}^{NT,+}(\widehat{z}_1,\widehat{z}_2) + \widehat{z}_2^2 w_{22}^{NT,+}(\widehat{z}_1,\widehat{z}_2) + O(\lambda). \end{aligned} \tag{4.28}$$

Let φ be defined and C^2 in a neighborhood of $(\widehat{z}_1,\widehat{z}_2)$, and suppose $\varphi \le w^+$ in this neighborhood and $\varphi(\widehat{z}_1,\widehat{z}_2) = w^+(\widehat{z}_1,\widehat{z}_2)$. Then $\nabla\varphi(\widehat{z}_1,\widehat{z}_2) = \nabla w^+(\widehat{z}_1,\widehat{z}_2)$ and the directional second derivative of φ in the $(\widehat{z}_1,\widehat{z}_2)$ directions satisfies

$$\begin{aligned} &\widehat{z}_1^2\varphi_{11}(\widehat{z}_1,\widehat{z}_2) + 2\widehat{z}_1\widehat{z}_2\varphi_{12}(\widehat{z}_1,\widehat{z}_2) + \widehat{z}_2^2\varphi_{22}(\widehat{z}_1,\widehat{z}_2) \\ &\le \max\begin{bmatrix} \widehat{z}_1^2 w_{11}^{S,+}(\widehat{z}_1,\widehat{z}_2) + 2\widehat{z}_1\widehat{z}_2 w_{12}^{S,+}(\widehat{z}_1,\widehat{z}_2) + \widehat{z}_2^2 w_{22}^{S,+}(\widehat{z}_1,\widehat{z}_2), \\ \widehat{z}_1^2 w_{11}^{NT,+}(\widehat{z}_1,\widehat{z}_2) + 2\widehat{z}_1\widehat{z}_2 w_{12}^{NT,+}(\widehat{z}_1,\widehat{z}_2) + \widehat{z}_2^2 w_{22}^{NT,+}(\widehat{z}_1,\widehat{z}_2) \end{bmatrix} \\ &= \widehat{z}_1^2 w_{11}^{NT,+}(\widehat{z}_1,\widehat{z}_2) + 2\widehat{z}_1\widehat{z}_2 w_{12}^{NT,+}(\widehat{z}_1,\widehat{z}_2) + \widehat{z}_2^2 w_{22}^{NT,+}(\widehat{z}_1,\widehat{z}_2) + K\lambda, \end{aligned}$$

where the constant K is independent of $(\widehat{z}_1,\widehat{z}_2) \in \partial S \cap \partial NT^+$.

Therefore,

$$\left(\mathcal{D}-\widetilde{\mathcal{U}}\right)(\varphi)(\widehat{z}_1,\widehat{z}_2) = \left(\mathcal{D}-\widetilde{\mathcal{U}}\right)(w^+)(\widehat{z}_1,\widehat{z}_2) - \frac{1}{2}(\sigma_1^2\widehat{z}_1^2 + \sigma_2^2\widehat{z}_2^2)K\lambda.$$

Recalling from Step 3 that $\left(\mathcal{D}-\widetilde{\mathcal{U}}\right)(w^+)(\widehat{z}_1,\widehat{z}_2) \ge MA\lambda + O(\lambda)$, where the $O(\lambda)$ tern is independent of $(\widehat{z}_1,\widehat{z}_2) \in \partial S \cap \partial NT^+$ we see that M can be chosen independently of $(\widehat{z}_1,\widehat{z}_2) \in \partial S \cap \partial NT^+$ and φ, so that $\left(\mathcal{D}-\widetilde{\mathcal{U}}\right)(\varphi)(\widehat{z}_1,\widehat{z}_2) \ge 0$.

Case 2: (z_1,z_2) is a point in the southern region.

From equation (4.27) we see that

$$pw^+ - z_1w_1^+ - z_2w_2^+ = \frac{1}{\lambda_2}w_2^+. \tag{4.29}$$

Moreover, by differentiating (4.27), we also get the system

$$\lambda_2 pw^+ + (-1-\lambda_2 z_2)w_2^+ - \lambda_2 z_1 w_1^+ = 0, \tag{4.30}$$
$$\lambda_2(p-1)w_1^+ + (-1-\lambda_2 z_2)w_{12}^+ - \lambda_2 z_1 w_{11}^+ = 0, \tag{4.31}$$
$$\lambda_2(p-1)w_2^+ + (-1-\lambda_2 z_2)w_{22}^+ - \lambda_2 z_1 w_{12}^+ = 0. \tag{4.32}$$

Adding $\frac{1-p}{\lambda_2}$ of equation (4.30), $\frac{z_1}{\lambda_2}$ of equation (4.31) and $\frac{z_2-\frac{1}{\lambda_2}}{\lambda_2}$ of equation (4.32) we get

$$-p(1-p)w^+ + 2(1-p)\sum_{i=1}^{2} z_i w_i^+ + z_1^2 w_{11}^+ + 2z_1 z_2 w_{12}^+ + z_2^2 w_{22}^+ = \frac{1}{\lambda_2^2} w_{22}^+. \tag{4.33}$$

Use (2.27) and the identity (A.36) that says that

$$\begin{aligned}
&\beta - p\left[r + \sum_{i=1}^{2}\mu_i z_i - \frac{(1-p)}{2}\left(\sigma_1^2 z_1^2 + \sigma_2^2 z_2^2 + 2\rho\sigma_1\sigma_2 z_1 z_2\right)\right] \\
&= (1-p)A + \frac{p(1-p)}{2}\left[\sigma_1^2(z_1-\theta_1)^2 + \sigma_2^2(z_2-\theta_2)^2 + 2\rho\sigma_1\sigma_2(z_1-\theta_1)(z_2-\theta_2)\right],
\end{aligned}$$

to compute

$$\begin{aligned}
&\mathcal{D}(w^+) - \widetilde{U}\left(pw^+ - z_1 w_1^+ - z_2 w_2^+\right) \\
&= \beta w^+ - \left(r + \sum_{i=1}^{2}\mu_i z_i\right)\frac{1}{\lambda_2}w_2^+ - \frac{1}{2}\left(z_1^2\sigma_1^2 + z_2^2\sigma_2^2\right)\frac{1}{\lambda_2^2}w_{22}^+ - \widetilde{U}\left(\frac{1}{\lambda_2}w_2^+\right).
\end{aligned} \tag{4.34}$$

Define $t \triangleq \frac{1}{\lambda_2}\left(\frac{1}{z_1} - \frac{1}{\widehat{z}_1}\right)$. Since in the southern region $0 < z_1 \leq \widehat{z}_1$, we see that $t \in [0,\infty]$. It also follows that

$$\frac{1}{z_1} = \frac{1}{\widehat{z}_1} + \lambda_2 t. \tag{4.35}$$

With this parametrization $t = 0$ corresponds to $(z_1, z_2) = (\widehat{z}_1, \widehat{z}_2)$, and $t = \infty$ corresponds to $(z_1, z_2) = \left(0, -\frac{1}{\lambda_2}\right)$. As we vary t (z_1, z_2) changes, but $(\widehat{z}_1, \widehat{z}_2)$ remain fixed.

From Proposition B.4, specifically (B.12) we get

$$\frac{z_2}{z_1} = \frac{1}{\lambda_2}\left(\frac{1+\lambda_2\widehat{z}_2}{\widehat{z}_1} - \frac{1}{z_1}\right) = \frac{\widehat{z}_2}{\widehat{z}_1} - t. \tag{4.36}$$

Divide equation (4.34) by z_1^p and use (4.35) and (4.36) to get

$$\begin{aligned}
&\frac{1}{z_1^p}\left(\mathcal{D}(w^+) - \widetilde{U}\left(pw^+ - z_1 w_1^+ - z_2 w_2^+\right)\right) \\
&= \frac{1}{z_1^p}\beta w^+ - \left(r\left(\frac{1}{\widehat{z}_1} + \lambda_2 t\right) + \mu_1 + \mu_2\left(\frac{\widehat{z}_2}{\widehat{z}_1} - t\right)\right)\frac{1}{\lambda_2 z_1^{p-1}}w_2^+ \\
&-\frac{1}{2}\left(\sigma_1^2 + \sigma_2^2\left(\frac{\widehat{z}_2}{\widehat{z}_1} - t\right)^2\right)\frac{1}{\lambda_2^2 z_1^{p-2}}w_{22}^+ - \widetilde{U}\left(\frac{1}{\lambda_2 z_1^{p-1}}w_2^+\right).
\end{aligned} \tag{4.37}$$

Note that $t = 0$ corresponds to $(z_1, z_2) = (\widehat{z}_1, \widehat{z}_2)$, and we have already shown that

$$\frac{1}{\widehat{z}_1^p}\left(\mathcal{D}(w^+) - \widetilde{U}\left(pw^+ - z_1 w_1^+ - z_2 w_2^+\right)\right)(\widehat{z}_1, \widehat{z}_2) \geq 0.$$

From Lemmas B.14 and B.24 we see that $\frac{1}{\lambda_2 z_1^{p-1}} w_2^+(z_1, z_2)$ and $\frac{1}{\lambda_2^2 z_1^{p-2}} w_{22}^+(z_1, z_2)$ depend on (z_1, z_2) only through $(\widehat{z}_1, \widehat{z}_2)$. We conclude that $\frac{\partial}{\partial t}\frac{1}{\lambda_2 z_1^{p-1}} w_2^+(z_1, z_2)$, $\frac{\partial}{\partial t}\frac{1}{\lambda_2^2 z_1^{p-2}} w_{22}^+(z_1, z_2) = 0$. It follows that $\frac{\partial}{\partial t}\widetilde{U}\left(\frac{1}{\lambda_2 z_1^{p-1}} w_2^+\right) = 0$.

Also note that the right hand side of (4.37) is a quadratic function of t, with positive coefficient of the t^2 term (see Lemma B.24). So it's enough to prove that

$$\frac{\partial}{\partial t}\left(\frac{1}{z_1^p}\mathcal{D}(w^+(z_1, z_2))\right) \geq 0, \text{ at } t = 0.$$

Finally for $\lambda > 0$ small enough

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}\left(\frac{1}{z_1^p}\mathcal{D}(w^+)(z_1, z_2)\right)\right)\Big|_{t=0} \\
&= \Big(-(r\lambda_2 - \mu_2)\frac{1}{\lambda_2 z_1^{p-1}} w_2^+ + \sigma_2^2\left(\frac{\widehat{z}_2}{\widehat{z}_1} - t\right)\frac{1}{\lambda_2^2 z_1^{p-2}} w_{22}^+\Big)\Big|_{t=0} \\
&= -\frac{(1-p)A^{p-1}}{\widehat{z}_1^{p-2}}\sigma_2^2\frac{\widehat{z}_2}{\widehat{z}_1} - \frac{A^{p-1}}{\widehat{z}_1^{p-1}}(-\mu_2 + r\lambda_2) + O\left(\lambda^{\frac{2}{3}}\right) \\
&= -\frac{A^{p-1}}{\widehat{z}_1^{p-1}}\left[-\mu_2 + (1-p)\sigma_2^2\theta_2\right] + \frac{\nu_2}{2}\sigma_2^2\frac{(1-p)A^{p-1}}{\widehat{z}_1^{p-1}}\lambda^{\frac{1}{3}} + O\left(\lambda^{\frac{2}{3}}\right) > 0,
\end{aligned}$$

where to get the second equality we have used the estimates from Lemmas B.14 and B.24, and to get the last equality, the fact that on the southern boundary of the NT^+ region $\widehat{z}_2 = \theta_2 - \frac{1}{2}\nu_2\lambda^{\frac{1}{3}} + O\left(\lambda^{\frac{2}{3}}\right)$.

Remark 4.19. The proof for the northern, eastern and western regions is similar and will be omitted for brevity.

The proof for the southwestern region follows by combining the proofs for southern and western regions. As in Remark B.9, fix $(z_1, z_2) \in SW$, and let (z_1^0, z_2^0) be the point of intersection of two lines: the first one connecting two points $(z_{1,SW}, z_{2,SW})$ and $(-\frac{1}{\lambda_1}, 0)$, and the second line connecting (z_1, z_2) with $(0, -\frac{1}{\lambda_2})$. Applying the analogous proof for the western region, we see that $(\mathcal{B}_1(w^+))(z_1^0, z_2^0) \geq 0$, and now applying it again for $\mathcal{B}_2$ operator, we find that

$$\left(\mathcal{B}_2(w^+)\right)(z_1, z_2) \geq \left(\mathcal{B}_1(w^+)\right)(z_1^0, z_2^0) \geq 0.$$

Similar argument shows that $\left(\mathcal{D} - \widetilde{\mathcal{U}}\right)(w^+)(z_1, z_2) \geq 0$ there.

□

4.6.4 Step 4: Summary

Case 1: (z_1, z_2) is (strictly inside) one of the nine regions.

In this case, in sections 4.4 and 4.6.1-4.6.3 we have shown that at (z_1, z_2)

$$\left(\mathcal{D} - \widetilde{\mathcal{U}}\right)(w^+),\ \mathcal{S}_2(w^+),\ \mathcal{B}_2(w^+),\ \mathcal{B}_1(w^+),\ \mathcal{S}_1(w^+) \geq 0$$

Thus inequality (4.6) holds, and from Lemma 4.7 it now follows that w^+ is supersolution.

Case 2: (z_1, z_2) is on one of boundaries of the NT^+ region.

Since $w^+ \in C^1(\mathcal{S}_u)$, we still have

$$\left(\mathcal{S}_i(\varphi)\right)(z_1, z_2), \left(\mathcal{B}_i(\varphi)\right)(z_1, z_2) \geq 0,\ i = 1, 2.$$

To conclude that assumption (4.6) holds, we need to show that $\left(\mathcal{D} - \widetilde{\mathcal{U}}\right)(w^+)(z_1, z_2) \geq 0$. Consider, for instance, (z_1, z_2) a point on the southern boundary of the NT^+ region. In sections 4.4 and 4.6.3 we have shown that both limits

$$\lim_{\substack{(\overline{z_1}, \overline{z_2}) \in S \\ (\overline{z_1}, \overline{z_2}) \to (z_1, z_2)}} \left(\mathcal{D} - \widetilde{\mathcal{U}}\right)(w^+)(\overline{z_1}, \overline{z_2}),$$

and

$$\lim_{\substack{(\underline{z_1}, \underline{z_2}) \in NT^+ \\ (\underline{z_1}, \underline{z_2}) \to (z_1, z_2)}} \left(\mathcal{D} - \widetilde{\mathcal{U}}\right)(w^+)(\underline{z_1}, \underline{z_2})$$

are positive. From Lemma 4.7 it now follows that w^+ is supersolution.

The proof for the other boundaries is the same.

□

4.7 Conclusion

We note that $w^\pm(\theta_1, \theta_2) = \frac{A^{p-1}}{p} - \gamma_2 \lambda^{\frac{2}{3}} \pm M\lambda$, and by Proposition B.11 $w^\pm|_{\partial \mathcal{S}_u} = 0$, so the Comparison Theorem 4.5 implies

$$\frac{A^{p-1}}{p} - \gamma_2 \lambda^{\frac{2}{3}} - M\lambda \leq u(\theta_1, \theta_2) \leq \frac{A^{p-1}}{p} - \gamma_2 \lambda^{\frac{2}{3}} + M\lambda.$$

This finishes the proof of the first main Theorem 4.1. □

Proof of Corollary 4.2: In the proof of Theorem 4.1 we constructed a supersolution w^+ and a subsolution w^- such that $w^+(z_1,z_1)-w^-(z_1,z_1)=O(\lambda)$, $w^\pm(z_1,z_1)=w^\pm(\theta_1,\theta_2)+O(\lambda)$ for fixed $(z_1,z_1)\in\mathcal{S}_u$. It follows that $u(z_1,z_1)=w^\pm(z_1,z_1)+O(\lambda)=w^\pm(\theta_1,\theta_2)+O(\lambda)=u(\theta_1,\theta_2)+O(\lambda)$.

Proof of Lemma 4.3: First note that $u_1^E(z_1,z_2)=O(\lambda)$ on the eastern boundary of the NT region. Here u_1^E refers to the limit of $u_1(z_1,z_2)$ evaluated when approaching the eastern boundary of NT from inside E. This follows because the value function u satisfies $\mathcal{S}_1(u)=0$ in E. A similar argument establishes that $u_1^W(z_1,z_2)=O(\lambda)$ on the western boundary of NT. Since u is concave, it follows that $u_1(z_1,z_2)=O(\lambda)$ everywhere inside NT region too. We make the same conclusion about $u_2(z_1,z_2)$.

The optimal rate of consumption is $v_x^{\frac{1}{p-1}}$ (see (2.20) and (2.16)), and the rate of consumption as a proportion of wealth is $\frac{1}{x}v_x^{\frac{1}{p-1}}$.Since $v(y_1,y_2,x)=x^pu\left(\frac{y_1}{x},\frac{y_2}{x}\right)$ (see (2.24)), for $\left(\frac{y_1}{x},\frac{y_2}{x}\right)\in NT$, we have

$$\begin{aligned}v_x(y_1,y_2,x) &= px^{p-1}u\left(\frac{y_1}{x},\frac{y_2}{x}\right)-\sum_{i=1}^{2}y_ix^{p-2}u_i\left(\frac{y_1}{x},\frac{y_2}{x}\right)\\ &= px^{p-1}u\left(\frac{y_1}{x},\frac{y_2}{x}\right)+O(\lambda).\end{aligned}$$

Therefore, Corollary 4.2 implies that the optimal consumption is

$$c=\left(A^{p-1}-p\gamma_2\lambda^{\frac{2}{3}}\right)^{\frac{1}{p-1}}+O(\lambda)=\left(A+\frac{p}{1-p}A^{2-p}\gamma_2\lambda^{\frac{2}{3}}\right)+O(\lambda). \tag{4.38}$$

5 Upper and Lower Bounds, when $\rho\neq 0$

We conclude by computing the value function in the remaining case when $\rho\neq 0$. We are not aware of a way, similar to before, to compute a heuristic expansion and then proceed by building sub- and supersolutions. Instead, we present an argument that builds upon the previous case to produce a subsolution and a tight upper bound, which together with Comparison Theorem 4.5 allow us again to estimate the value function up order $O(\lambda)$. To do that, an auxiliary problem is introduced, where the futures contract of type one is identical, and the futures contract of type two can be replicated by buying one original futures contract of type two, and additionally buying $\rho\frac{\sigma_2}{\sigma_1}$ number of original futures contracts of type one. In Lemma 5.5 we show that the new value function, corresponding to the auxiliary problem, defined in (2.11) with the parameters of the auxiliary problem dominates the original value function. The intuitive explanation is that in the original model the futures contracts are replicable using the contracts from the auxiliary model, and we will construct the transaction costs in the auxiliary model, so that the cost of replication would not be higher than in the original model. We now proceed to execute this plan.

We had defined previously the transaction costs for trading in futures of type i as $\lambda_= \alpha_i \lambda$. Without loss of generality we can assume that $\alpha_2\mu_1 \geq \alpha_1\mu_2$. However, for the rest of this section, we will have to restricts the parameters further:

Assumption 5.1.

(i) The correlation ρ is strictly positive. The case of $\rho = 0$ has already been discussed, and the case $\rho < 0$ can be handled similarly to this.

(ii) Assume that $\alpha_2\mu_1 > \alpha_1\mu_2$. We have already assumed that $\alpha_2\mu_1 \geq \alpha_1\mu_2$, but we need to further exclude the case when $\alpha_2\mu_1 = \alpha_1\mu_2$.

(iii) The Merton optimal proportion (θ_1, θ_2) is in the first quadrant, i.e., $\theta_i > 0,\ i = 1, 2$.

(iv) $\alpha_2 - \rho\frac{\sigma_2}{\sigma_1}\alpha_1 > 0$.

Recall our model (2.1), (2.2)-(2.3)

$$\begin{aligned} F_i(t) &= F_i(0) + \mu_i t + \sigma_i B_i(t), && (5.1)\\ dX(t) &= \sum_{i=1}^{2} Y_i(t)dF_i(t) - \sum_{i=1}^{2} \lambda_i(dL_i(t) + dM_i(t)) + (rX(t) - C(t))dt, && (5.2)\\ dY_i(t) &= dL_i(t) - dM_i(t). && (5.3) \end{aligned}$$

Consider the following change of variables

$$\tilde{F}_1 = F_1, \quad \tilde{F}_2 = F_2 - \rho\frac{\sigma_2}{\sigma_1}F_1. \tag{5.4}$$

Our goal is use this change of variables to define an auxiliary problem

$$\left\{ \begin{array}{lcl} \tilde{y}_2 &=& y_2, \\ \tilde{y}_1 &=& y_1 + \rho\frac{\sigma_2}{\sigma_1}y_2. \end{array} \right. \tag{5.5}$$

so that

$$\left\{ \begin{array}{lcl} y_2 &=& \tilde{y}_2, \\ y_1 &=& \tilde{y}_1 - \rho\frac{\sigma_2}{\sigma_1}\tilde{y}_2. \end{array} \right. \tag{5.6}$$

In the reduced two dimensional space this change will be

$$\left\{ \begin{array}{lcl} \tilde{z}_2 &=& z_2, \\ \tilde{z}_1 &=& z_1 + \rho\frac{\sigma_2}{\sigma_1}z_2. \end{array} \right. \tag{5.7}$$

and

$$\left\{ \begin{array}{lcl} z_2 &=& \tilde{z}_2, \\ z_1 &=& \tilde{z}_1 - \rho\frac{\sigma_2}{\sigma_1}\tilde{z}_2. \end{array} \right. \tag{5.8}$$

In order to be able to write $\tilde{F}_i(t) = \tilde{F}_i(0) + \tilde{\mu}_i t + \tilde{\sigma}_i \tilde{B}_i(t)$ in a similar form to our original model (5.1), the following change of parameters is needed

$$\begin{aligned} &\tilde{\sigma}_1 = \sigma_1, \quad \tilde{\sigma}_2 = \sigma_2\sqrt{1-\rho^2}, \quad \tilde{\rho} = 0, \\ &\tilde{\mu}_1 = \mu_1, \quad \tilde{\mu}_2 = \mu_2 - \rho\frac{\sigma_2}{\sigma_1}\mu_1, \\ &\tilde{\alpha}_1 = \alpha_1, \quad \tilde{\alpha}_2 = \alpha_2 - \rho\frac{\sigma_2}{\sigma_1}\alpha_1, \\ &\tilde{\lambda}_i = \tilde{\alpha}_i\lambda. \end{aligned} \tag{5.9}$$

Note by Assumption 5.1.(iv) $\tilde{\alpha}_2 > 0$. Under this change of variables, we have

$$\begin{aligned} \tilde{\theta}_2 &= \frac{\tilde{\mu}_2}{(1-p)\tilde{\sigma}_2^2} = \frac{\left(\mu_2 - \rho\frac{\sigma_2}{\sigma_1}\mu_1\right)\sigma_1}{(1-p)\sigma_2^2(1-\rho^2)\sigma_1} = \theta_2, \\ \tilde{\theta}_1 - \rho\frac{\sigma_2}{\sigma_1}\tilde{\theta}_2 &= \frac{\mu_1}{(1-p)\sigma_1^2} - \rho\frac{\sigma_2}{\sigma_1}\frac{\mu_2\sigma_1 - \rho\sigma_2\mu_1}{(1-p)\sigma_2^2\sigma_1(1-\rho^2)} = \theta_1. \end{aligned}$$

Remark 5.2. From Assumption 5.1.(iii) $\theta_1, \theta_2 > 0$, so we conclude that $\tilde{\theta}_1, \tilde{\theta}_2 > 0$.

Similarly to (3.51) and (3.52) define

$$\tilde{\nu}_i = \sqrt[3]{12\tilde{\alpha}_i\tilde{\theta}_i^2\frac{\tilde{\sigma}_1^2\tilde{\theta}_1^2 + \tilde{\sigma}_2^2\tilde{\theta}_2^2}{(1-p)\tilde{\sigma}_i^2}}, \tag{5.10}$$

$$\tilde{\gamma}_2 = A^{p-2}\sum_{i=1}^{2}\sqrt[3]{\frac{9}{32}\tilde{\alpha}_i^2(1-p)\tilde{\sigma}_i^2\tilde{\theta}_i^4\left(\tilde{\sigma}_1^2\tilde{\theta}_1^2 + \tilde{\sigma}_2^2\tilde{\theta}_2^2\right)^2}. \tag{5.11}$$

Remark 5.3. From Assumption 5.1.(ii) it follows that $\tilde{\alpha}_2\tilde{\mu}_1 > \tilde{\alpha}_1\tilde{\mu}_2$. It follows that $\tilde{\theta}_1\sqrt[3]{\tilde{\alpha}_2\frac{\tilde{\theta}_2^2}{\tilde{\sigma}_2^2}} > \tilde{\theta}_2\sqrt[3]{\tilde{\alpha}_1\frac{\tilde{\theta}_1^2}{\tilde{\sigma}_1^2}}$, which is equivalent to $\tilde{\nu}_2\tilde{\theta}_1 > \tilde{\nu}_1\tilde{\theta}_2$, which in turn is equivalent to

$$\frac{\tilde{\theta}_2 + \frac{1}{2}\tilde{\nu}_2\lambda^{\frac{1}{3}} + O\left(\lambda^{\frac{2}{3}}\right)}{\tilde{\theta}_1 + \frac{1}{2}\tilde{\nu}_1\lambda^{\frac{1}{3}} + O\left(\lambda^{\frac{2}{3}}\right)} > \frac{\tilde{\theta}_2 - \frac{1}{2}\tilde{\nu}_2\lambda^{\frac{1}{3}} + O\left(\lambda^{\frac{2}{3}}\right)}{\tilde{\theta}_1 - \frac{1}{2}\tilde{\nu}_1\lambda^{\frac{1}{3}} + O\left(\lambda^{\frac{2}{3}}\right)}.$$

Hence the slope of radial line passing through $\left(\tilde{\theta}_1 + \frac{1}{2}\tilde{\nu}_1\lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}}), \tilde{\theta}_2 + \frac{1}{2}\tilde{\nu}_2\lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}})\right)$ is greater then the slope of the radial line passing through $\left(\tilde{\theta}_1 - \frac{1}{2}\tilde{\nu}_1\lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}}), \tilde{\theta}_2 - \frac{1}{2}\tilde{\nu}_2\lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}})\right)$

□

Theorem 5.4. Assume that $0 < p < 1,\ A > 0$, and assume Assumptions 5.1 hold. Then the value function u satisfies

$$u(\theta_1, \theta_2) = \frac{A^{p-1}}{p} - A^{p-2}\lambda^{\frac{2}{3}}\sum_{i=1}^{2}\sqrt[3]{\frac{9}{32}\tilde{\alpha}_i^2(1-p)\tilde{\sigma}_i^2\tilde{\theta}_i^4\left(\tilde{\sigma}_1^2\tilde{\theta}_1^2 + \tilde{\sigma}_2^2\tilde{\theta}_2^2\right)^2} + O(\lambda). \tag{5.12}$$

We divide the proof of this theorem into two parts. Lemma 5.5 will be used to find a tight upper bound, which would end up being a supersolution from the case when $\rho = 0$. The second step will be the construction of subsolution. Together with Comparison Theorem 4.5 this will give us (5.12).

Lemma 5.5. Assume that $0 < p < 1,\ A > 0$, and assume Assumptions 5.1 hold. Consider the change of variables (5.5)-(5.8), and let $\tilde{u}(\tilde{z}_1, \tilde{z}_2) = \tilde{v}(\tilde{y}_1, \tilde{y}_2, 1)$ be the value functions associated with the auxiliary problem, for which the futures contracts are independent and their evolution given by (5.4). Then $v(y_1, y_2, x) \leq \tilde{v}(\tilde{y}_1, \tilde{y}_2, x)$ and thus also $u(z_1, z_2) \leq \tilde{u}(\tilde{z}_1, \tilde{z}_2)$.

Proof: The solvency region for the auxiliary problem is defined as

$$\widetilde{\mathcal{S}_v} = \left\{ (\tilde{y}_1, \tilde{y}_2, x) | x - \tilde{\lambda}_1 |\tilde{y}_1| - \tilde{\lambda}_2 |\tilde{y}_2| > 0 \right\}.$$

We can similarly define $\widetilde{\mathcal{S}_u} = \left\{ (\tilde{z}_1, \tilde{z}_2) | 1 - \tilde{\lambda}_1 |\tilde{z}_1| - \tilde{\lambda}_2 |\tilde{z}_2| > 0 \right\}$. The reader can then verify that for any $(y_1, y_2, x) \in \mathcal{S}_v$, we have that $(\tilde{y}_1, \tilde{y}_2, x)$ defined by (5.5) is in $\widetilde{\mathcal{S}_v}$, and similarly $(z_1, z_2) \in \mathcal{S}_u$, we have that $(\tilde{z}_1, \tilde{z}_2)$ is in $\widetilde{\mathcal{S}_u}$.

We call a policy $(l_i, m_i, c)_{i=1,2}$ admissible in the auxiliary problem, if $(Y_1(t), Y_2(t), X(t))$, given by (2.6), (2.7) with the parameters of the auxiliary problem instead, is in the closure of $\widetilde{\mathcal{S}_v}$ for all $t \geq 0$. We denote by $\tilde{\mathcal{A}}(\tilde{y}_1, \tilde{y}_2, x)$ the set of all such policies. As before, we will abuse notation and also write $(L_i, M_i, C)_{i=1,2} \in \tilde{\mathcal{A}}(\tilde{y}_1, \tilde{y}_2, x)$.

The auxiliary problem is an optimization problem (2.11) but with tilde parameters given by (5.9). That is, for $(\tilde{y}_1, \tilde{y}_2, x) \in \widetilde{\mathcal{S}_v}$

$$v(\tilde{y}_1, \tilde{y}_2, x) \triangleq \sup_{c, l_i, m_i \in \tilde{\mathcal{A}}(\tilde{y}_1, \tilde{y}_2, x)} E \left[\int_0^\infty e^{-\beta t} U(X(t) c(t)) dt \right].$$

Given an admissible policy $(L_i, M_i, C)_{i=1,2} \in \mathcal{A}(y_1, y_2, x)$, we construct a policy for the initial position $(\tilde{y}_1, \tilde{y}_2, x)$ defined in (5.5), as follows

$$\tilde{L}_2(t) = L_2(t), \quad \tilde{L}_1(t) = L_1(t) + \rho \frac{\sigma_2}{\sigma_1} L_2(t), \tag{5.13}$$

$$\tilde{M}_2(t) = M_2(t), \quad \tilde{M}_1(t) = M_1(t) + \rho \frac{\sigma_2}{\sigma_1} M_2(t). \tag{5.14}$$

Moreover, we define

$$\tilde{Y}_2(t) = Y_2(t), \quad \tilde{Y}_1(t) = Y_1(t) + \rho \frac{\sigma_2}{\sigma_1} Y_2(t), \tag{5.15}$$

$$\tilde{B}_1(t) = B_1(t), \quad \tilde{B}_2(t) = \frac{1}{\sqrt{1-\rho^2}} \left(B_2(t) - \rho B_1(t) \right). \tag{5.16}$$

Then $\tilde{B}_1(t), \tilde{B}_2(t)$ are uncorrelated Brownian motions; thus indeed $\tilde{\rho} = 0$. The reader can also verify that the following holds

$$\tilde{Y}_i(t) = \tilde{L}_i(t) - \tilde{M}_i(t).$$
$$\sum_{i=1}^{2} \tilde{\lambda}_i(\tilde{L}_i + \tilde{M}_i) = \sum_{i=1}^{2} \lambda_i(L_i + M_i),$$
$$\sum_{i=1}^{2} \tilde{\mu}_i \tilde{Y}_i = \sum_{i=1}^{2} \mu_i Y_i,$$
$$B_1(t) = \tilde{B}_1(t), \quad B_2(t) = \sqrt{1-\rho^2}\tilde{B}_2(t) + \rho \tilde{B}_1(t).$$

And also

$$dX(t) = \sum_{i=1}^{2} \tilde{Y}_i(t) d\tilde{F}_i(t) - \sum_{i=1}^{2} \tilde{\lambda}_i(d\tilde{L}_i(t) + d\tilde{M}_i(t)) + (rX(t) - C(t))dt, \tag{5.17}$$
$$d\tilde{Y}_i(t) = d\tilde{L}_i(t) - d\tilde{M}_i(t). \tag{5.18}$$

Then $(\tilde{Y}_1(t), \tilde{Y}_2(t), X(t))$ given by (5.17), (5.18) is in the closure of $\widetilde{\mathcal{S}_v}$ for all $t \geq 0$. We conclude that we have constructed an admissible policy $(\tilde{L}_i, \tilde{M}_i, C)_{i=1,2} \in \mathcal{A}(\tilde{y}_1, \tilde{y}_2, x)$.

To summarize, for any initial position $(x, y_1, y_2) \in \mathcal{S}_v$, we have defined an auxiliary problem with parameters (5.9), and given an admissible policy $(L_i, M_i, C)_{i=1,2} \in \mathcal{A}(y_1, y_2, x)$, we have constructed an admissible policy $(\tilde{L}_i, \tilde{M}_i, C)_{i=1,2} \in \mathcal{A}(\tilde{y}_1, \tilde{y}_2, x)$ that has the same consumption. Hence the value function of the auxiliary problem with the initial position $(x, \tilde{y}_1, \tilde{y}_2)$ dominates the value function of the original problem $v(x, y_1, y_2) \leq \tilde{v}(x, \tilde{y}_1, \tilde{y}_2)$.

We conclude that for any $(z_1, z_2) \in \mathcal{S}_u$,

$$u(z_1, z_2) = v(1, z_1, z_2) \leq \tilde{v}(1, \tilde{z}_1, \tilde{z}_2,) = \widetilde{u}(\tilde{z}_1, \tilde{z}_2).$$

$\square$

Remark 5.6. For $(\tilde{y}_1, \tilde{y}_2, x) \in \tilde{\mathcal{S}}_v$ consider an admissible policy $(\tilde{L}_i, \tilde{M}_i, C)_{i=1,2} \in \mathcal{A}(\tilde{y}_1, \tilde{y}_2, x)$. Note that this policy may be improved. For example, it's possible that both $L_2(t), M_1(t)$ are increasing at the same time t. Causing both $\tilde{L}_1(t), \tilde{M}_1(t)$, to simultaneously increase, which corresponds to both buying and selling futures contract of type one in the auxiliary problem at the same time, and paying double the transaction costs. By Hahn Decomposition Theorem there exist two non-decreasing, right-continuous processes $\widetilde{\widetilde{L_1}}$ and $\widetilde{\widetilde{M_1}}$ such that $d\widetilde{Y_1}(t) = \widetilde{\widetilde{L_1}}(t) - \widetilde{\widetilde{M_1}}(t)$, and $\widetilde{\widetilde{L_1}}$ and $\widetilde{\widetilde{M_1}}$ cannot increase at the same time, i.e., $d\widetilde{\widetilde{L_1}} d\widetilde{\widetilde{M_1}} = 0$. In other words, this is the minimal decomposition. It follows that $\widetilde{\widetilde{L_1}} \leq \widetilde{L_1}$ and $\widetilde{\widetilde{M_1}} \leq \widetilde{M_1}$. We conclude that

$$\sum_{i=1}^{2} \tilde{\lambda}_i (d\tilde{\tilde{L}}_i(t) + d\widetilde{\widetilde{M}}_i(t)) \le \sum_{i=1}^{2} \lambda_i (dL_i(t) + dM_i(t)).$$

It follows that $\tilde{X}(t) \ge X(t)$, where

$$d\tilde{X}(t) = \sum_{i=1}^{2} \widetilde{Y}_i(t)(\tilde{\mu}_i dt + \tilde{\sigma}_i d\tilde{B}_i(t)) - \sum_{i=1}^{2} \tilde{\lambda}_i (d\tilde{\tilde{L}}_i(t) + d\widetilde{\widetilde{M}}_i(t)) + (r\tilde{X}(t) - C(t))dt$$

Thus the tilde policy $(\tilde{\tilde{L}}_i, \widetilde{\widetilde{M}}_i, C)_{i=1,2} \in \mathcal{A}(\tilde{y}_1, \tilde{y}_2, x)$ is still admissible.

Remark 5.7. The intuition behind the construction of subsolution is the following. It was mentioned before that a futures contract of type one is identical in both the original and the auxiliary problems. Also, buying (selling) θ_1 amount of futures contract of type one and θ_2 amount of futures contract of type two in the original problem costs the same as buying (selling) $\tilde{\theta}_1$ of futures contract of type one and $\tilde{\theta}_2$ of futures contract of type two in the auxiliary problem. This is because $\sum_{i=1}^{2} \lambda_i \theta_i = \sum_{i=1}^{2} \tilde{\lambda}_i \tilde{\theta}_i$. It follows that if we could construct a region where we only need to trade futures contract of type one by itself, or when trading both futures contracts, doing it only in quantities proportional to (θ_1, θ_2), we could use the construction of subsolution from the previous case when $\rho = 0$. This is the intuition behind the Definition 5.9.

5.1 The NT^- region and other sub-regions of $\mathcal{S}_u$.

Fix a positive constant B. We define functions

$$\tilde{h}_i(\delta) = \tilde{\alpha}_i \left(\frac{3}{2} \delta^2 \lambda^{\frac{2}{3}} - \frac{1}{\tilde{\nu}_i^2} \delta^4 + \frac{3}{2} B \delta^2 \lambda^{\frac{4}{3}} \right),$$

$$
\begin{aligned}
\tilde{f}_N^-(\delta_1,\delta_2) &= \lambda_2 - p\tilde{\gamma}_2 A^{1-p}\lambda_2\lambda^{\frac{2}{3}} - pMA^{1-p}\lambda_2\lambda - p\lambda_2\sum_{i=1}^{2}\frac{\tilde{h}_i(\delta_i)}{\tilde{\nu}_i} \\
&\quad -(1-\lambda_2(\delta_2+\tilde{\theta}_2))\frac{\tilde{h}_2'(\delta_2)}{\tilde{\nu}_2} + \left(\lambda_2(\delta_1+\tilde{\theta}_1) - \rho\frac{\sigma_2}{\sigma_1}\right)\frac{\tilde{h}_1'(\delta_1)}{\tilde{\nu}_1}, \quad (5.19)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{f}_S^-(\delta_1,\delta_2) &= \lambda_2 - p\tilde{\gamma}_2 A^{1-p}\lambda_2\lambda^{\frac{2}{3}} - pMA^{1-p}\lambda_2\lambda - p\lambda_2\sum_{i=1}^{2}\frac{\tilde{h}_i(\delta_i)}{\tilde{\nu}_i} \\
&\quad +(1+\lambda_2(\delta_2+\tilde{\theta}_2))\frac{\tilde{h}_2'(\delta_2)}{\tilde{\nu}_2} + \left(\lambda_2(\delta_1+\tilde{\theta}_1) + \rho\frac{\sigma_2}{\sigma_1}\right)\frac{\tilde{h}_1'(\delta_1)}{\tilde{\nu}_1}, \quad (5.20)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{f}_W^-(\delta_1,\delta_2) &= \lambda_1 - p\tilde{\gamma}_2 A^{1-p}\lambda_1\lambda^{\frac{2}{3}} - pMA^{1-p}\lambda_1\lambda - p\lambda_1\sum_{i=1}^{2}\frac{\tilde{h}_i(\delta_i)}{\tilde{\nu}_i} \\
&\quad +(1+\lambda_1(\delta_1+\tilde{\theta}_1))\frac{\tilde{h}_1'(\delta_1)}{\tilde{\nu}_1} + \lambda_1(\delta_2+\tilde{\theta}_2)\frac{\tilde{h}_2'(\delta_2)}{\tilde{\nu}_2}, \quad (5.21)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{f}_E^-(\delta_1,\delta_2) &= \lambda_1 - p\tilde{\gamma}_2 A^{1-p}\lambda_1\lambda^{\frac{2}{3}} - pMA^{1-p}\lambda_1\lambda - p\lambda_1\sum_{i=1}^{2}\frac{\tilde{h}_i(\delta_i)}{\tilde{\nu}_i} \\
&\quad -(1-\lambda_1(\delta_1+\tilde{\theta}_1))\frac{\tilde{h}_1'(\delta_1)}{\tilde{\nu}_1} + \lambda_1(\delta_2+\tilde{\theta}_2)\frac{\tilde{h}_2'(\delta_2)}{\tilde{\nu}_2}, \quad (5.22)
\end{aligned}
$$

From Remark 4.9 we have that for any $\delta_2 \in [-\frac{1}{2}\tilde{\nu}_2\lambda^{\frac{1}{3}}, \frac{1}{2}\tilde{\nu}_2\lambda^{\frac{1}{3}}]$ there exists a C^2 function $\tilde{\delta}_{W,1}^-(\delta_2) = -\frac{1}{2}\tilde{\nu}_1\lambda^{\frac{1}{3}}(1-\tilde{\xi}_W^-(\delta_2)\lambda^{\frac{1}{3}})$ such that $\tilde{f}_W^-(\tilde{\delta}_{W,1}^-(\delta_2),\delta_2) = 0$. We have similar result for the zero level of $\tilde{f}_E^-$.

Remark 5.8. Note that

$$
\tilde{f}_S^-(\delta_1,\delta_2) = \rho\frac{\sigma_2}{\sigma_1}\tilde{f}_W^-(\delta_1,\delta_2) + \tilde{\alpha}_2\left[\lambda - p\tilde{\gamma}_2 A^{1-p}\lambda^{\frac{5}{3}} + \frac{\delta_2}{\tilde{\nu}_2}\left(3\lambda^{\frac{2}{3}} - \frac{4}{\tilde{\nu}_2^2}\delta_2^2 + 3B\lambda^{\frac{4}{3}}\right)\right] + O(\lambda^2). \quad (5.23)
$$

Let $\delta_i = -\frac{1}{2}\tilde{\nu}_i\lambda^{\frac{1}{3}}(1-\xi_i\lambda^{\frac{1}{3}}),\ i = 1,2$ where $\xi_i = \sqrt{\xi^2+\eta_i}$, with ξ defined previously as $\xi = \sqrt{\frac{2}{3}p\tilde{\gamma}_2 A^{1-p} + B} > 0$. From Remark 4.10 we see that

$$
\tilde{f}_W^-(\delta_1,\delta_2) = \frac{3}{2}\tilde{\alpha}_1\eta_1\lambda^{\frac{5}{3}} + O(\lambda^2),
$$

$$
\tilde{f}_S^-(\delta_1,\delta_2) = \frac{3}{2}\rho\frac{\sigma_2}{\sigma_1}\tilde{\alpha}_1\eta_1\lambda^{\frac{5}{3}} + \frac{3}{2}\tilde{\alpha}_2\eta_2\lambda^{\frac{5}{3}} + O(\lambda^2).
$$

If $\eta_1,\eta_2 > 0$, then $\tilde{f}_W^-(\delta_1,\delta_2), \tilde{f}_S^-(\delta_1,\delta_2) > 0$. If $\eta_1,\eta_2 < 0$, then $\tilde{f}_W^-(\delta_1,\delta_2), \tilde{f}_S^-(\delta_1,\delta_2) < 0$. If $\eta_1 > 0$ and $\eta_2 < 0$ big enough, then $\tilde{f}_W^-(\delta_1,\delta_2) > 0$ and $\tilde{f}_S^-(\delta_1,\delta_2) < 0$. Finally, if $\eta_1 < 0$ and $\eta_2 > 0$ big enough, then $\tilde{f}_W^-(\delta_1,\delta_2) < 0$ and $\tilde{f}_S^-(\delta_1,\delta_2) > 0$. Thus by continuity there exist a point $(\tilde{\delta}_{1,SW}^-, \tilde{\delta}_{2,SW}^-)$ such that

$$
\tilde{f}_S^-(\tilde{\delta}_{1,SW}^-,\tilde{\delta}_{2,SW}^-) = 0 = \tilde{f}_W^-(\tilde{\delta}_{1,SW}^-,\tilde{\delta}_{2,SW}^-).
$$

Furthermore, $(\tilde{\delta}^-_{1,SW}, \tilde{\delta}^-_{2,SW}) = \Big(-\frac{1}{2}\tilde{\nu}_1\lambda^{\frac{1}{3}}(1-\xi\lambda^{\frac{1}{3}}) + O(\lambda), -\frac{1}{2}\tilde{\nu}_2\lambda^{\frac{1}{3}}(1-\xi\lambda^{\frac{1}{3}}) + O(\lambda)\Big)$.

Similar logic establishes the existence of another point $(\tilde{\delta}^-_{1,NE}, \tilde{\delta}^-_{2,NE})$ such that

$$\tilde{f}^-_N(\tilde{\delta}^-_{1,NE}, \tilde{\delta}^-_{2,NE}) = 0 = \tilde{f}^-_E(\tilde{\delta}^-_{1,NE}, \tilde{\delta}^-_{2,NE}),$$

with $(\tilde{\delta}^-_{1,NE}, \tilde{\delta}^-_{2,NE}) = \Big(\frac{1}{2}\tilde{\nu}_1\lambda^{\frac{1}{3}}(1-\xi\lambda^{\frac{1}{3}}) + O(\lambda), \frac{1}{2}\tilde{\nu}_2\lambda^{\frac{1}{3}}(1-\xi\lambda^{\frac{1}{3}}) + O(\lambda)\Big)$.

□

Definition 5.9. Define the location of northeastern and southwestern corners in the tilde variables respectively as

$$(\tilde{z}^-_{1,NE}, \tilde{z}^-_{2,NE}) = (\tilde{\delta}^-_{1,NE}, \tilde{\delta}^-_{2,NE}) + (\tilde{\theta}_1, \tilde{\theta}_2), \tag{5.24}$$
$$(\tilde{z}^-_{1,SW}, \tilde{z}^-_{2,SW}) = (\tilde{\delta}^-_{1,SW}, \tilde{\delta}^-_{2,SW}) + (\tilde{\theta}_1, \tilde{\theta}_2) \tag{5.25}$$

Remark 5.10.

$$\begin{aligned}\tilde{f}^-_W(\delta_1, \delta_2) &= \lambda_1 - p\tilde{\gamma}_2 A^{1-p}\lambda_1\lambda^{\frac{2}{3}} + \frac{\tilde{h}'_1(\delta_1)}{\tilde{\nu}_1} + O(\lambda^2) \\ &= \lambda_1 - p\tilde{\gamma}_2 A^{1-p}\lambda_1\lambda^{\frac{2}{3}} + \frac{\tilde{\alpha}_1\delta_1}{\tilde{\nu}_1}\left(3\lambda^{\frac{2}{3}} - \frac{4}{\tilde{\nu}_1^2}\delta_1^2 + 3B\lambda^{\frac{4}{3}}\right) + O(\lambda^2)\end{aligned}$$

It's easy to see that $\tilde{f}^-_W(\frac{1}{2}\tilde{\nu}_1\lambda^{\frac{1}{3}}, \delta_2) > 0$ and $\tilde{f}^-_W(-\frac{1}{2}\tilde{\nu}_1\lambda^{\frac{1}{3}}, \delta_2) < 0$, for any $\delta_2 = O(\lambda^{\frac{1}{3}})$.

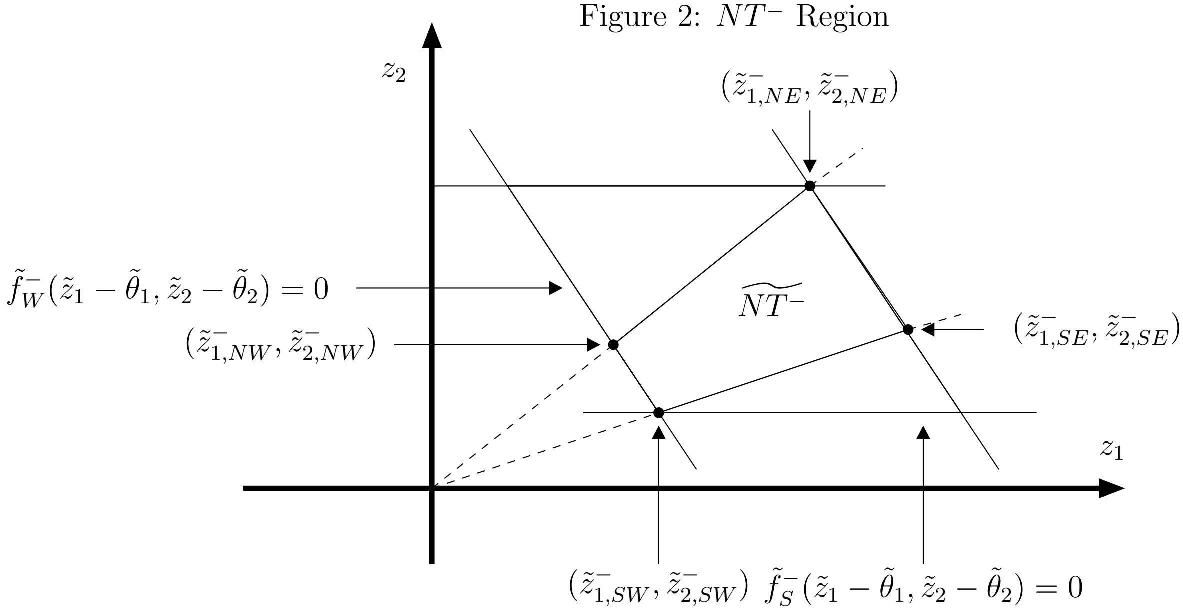
Consider the radial line passing through the northeastern corner $(\tilde{z}^-_{1,NE}, \tilde{z}^-_{2,NE})$ and the origin. The point $(\tilde{\theta}_1, \tilde{\theta}_2) + (-\frac{1}{2}\tilde{\nu}_1\lambda^{\frac{1}{3}}, \delta_2)$ lies on this line for a suitable choice of $\delta_2 = O(\lambda^{\frac{1}{3}})$. Similarly for another value of $\delta_2 = O(\lambda^{\frac{1}{3}})$ the point $(\tilde{\theta}_1, \tilde{\theta}_2) + (\frac{1}{2}\tilde{\nu}_1\lambda^{\frac{1}{3}}, \delta_2)$ is on this line too. By continuity of $\tilde{f}^-_W$, it follows from Lemma 4.8 that this line has an intersection point the curve $\tilde{f}^-_W(\tilde{z}_1 - \tilde{\theta}_1, \tilde{z}_2 - \tilde{\theta}_2) = 0$. We call this point the northwestern corner, and refer to it as $(\tilde{z}^-_{1,NW}, \tilde{z}^-_{2,NW})$. Furthermore, from Remark 5.3 it follows that $\tilde{z}^-_{2,SW} < \tilde{z}^-_{2,NW}$.

The southeastern corner $(\tilde{z}^-_{1,SE}, \tilde{z}^-_{2,SE})$ in the tilde variables is defined to be a points on the radial line passing through $(\tilde{z}^-_{1,SW}, \tilde{z}^-_{2,SW})$ and satisfying $\tilde{f}^-_E(\tilde{z}^-_{1,SW} - \tilde{\theta}_1, \tilde{z}^-_{2,SW} - \tilde{\theta}_2)$.

□

We also define the northeastern, southeastern, southwestern and the northwestern corners $(z^-_{1,NE}, z^-_{2,NE})$, $(z^-_{1,SE}, z^-_{2,SE})$, $(z^-_{1,SW}, z^-_{2,SW})$ and $(z^-_{1,NW}, z^-_{2,NW})$ respectively, by changing the variables using (5.8). Also note that the southeastern and the southwestern corners $(z^-_{1,SE}, z^-_{2,SE})$, $(z^-_{1,SW}, z^-_{2,SW})$ are still on the same radial line, since the change of variables (5.8) is linear with the origin as a fixed point. The same holds for $(z^-_{1,NE}, z^-_{2,NE})$ and $(z^-_{1,NW}, z^-_{2,NW})$.

Figure 2: $\widetilde{NT^-}$ Region



Definition 5.11. We define the NT^- region for the subsolution, as the region with corners $(z^-_{1,SW}, z^-_{2,SW})$, $(z^-_{1,SE}, z^-_{2,SE})$, $(z^-_{1,NW}, z^-_{2,NW})$ and $(z^-_{1,NE}, z^-_{2,NE})$ and with western boundary defined by $(\tilde{\delta}^-_{W,1}(\delta_2) - \rho\frac{\sigma_2}{\sigma_1}\delta_2 + \theta_1, \delta_2 + \theta_2)$, for $\delta_2 \in [\tilde{\delta}^-_{2,SW}, \tilde{\delta}^-_{2,NW}]$, and the eastern boundary defined similarly. The northern boundary is a straight line connecting the two corners $(z^-_{1,NW}, z^-_{2,NW})$ and $(z^-_{1,NE}, z^-_{2,NE})$, and the southern boundary is defined similarly. We also define the $\widetilde{NT^-}$ region by

$$(\tilde{z}_1, \tilde{z}_2) \in \widetilde{NT^-} \Leftrightarrow (\tilde{z}_1 - \rho\frac{\sigma_2}{\sigma_1}\tilde{z}_2, \tilde{z}_2) \in NT^- \tag{5.26}$$

Definition 5.12. We divide the solvency region $\mathcal{S}_u$ into NT^-, and the northern, southern, eastern, western, northeastern, southeastern, northwestern and southwestern regions, by connecting with straight lines each of the four corners of the NT^- region with two appropriate corners of the solvency region. That is, we connect the northeastern corner $(z^-_{1,NE}, z^-_{2,NE})$ with $(0, \frac{1}{\lambda_2})$ and $(\frac{1}{\lambda_1}, 0)$, the northwestern corner $(z^-_{1,NW}, z^-_{2,NW})$ corner with $(0, \frac{1}{\lambda_2})$ and $(-\frac{1}{\lambda_1}, 0)$, the southwestern corner $(z^-_{1,SW}, z^-_{2,SW})$ with $(0, -\frac{1}{\lambda_2})$ and $(-\frac{1}{\lambda_1}, 0)$ and the southeastern corner $(z^-_{1,SE}, z^-_{2,SE})$ with $(0, -\frac{1}{\lambda_2})$ and $(\frac{1}{\lambda_1}, 0)$. We have divided the solvency region $\mathcal{S}_u$ into nine regions the same way it was done before in Definition 4.16.

5.2 Construction of candidate subsolution.

Define, for $(\tilde z_1, \tilde z_2) \in \widetilde{NT^-}$

$$\tilde w^-(\tilde z_1, \tilde z_2) = \frac{A^{p-1}}{p} - \tilde\gamma_2\, \lambda^{\frac{2}{3}} - M\lambda - \sum_{i=1}^{2} \frac{A^{p-1}}{\tilde\nu_i}\tilde h_i(\tilde z_i - \tilde\theta_i), \tag{5.27}$$

For $(z_1, z_2) \in \overline{NT^-}$, define also

$$w^-(z_1, z_2) = w^-(\tilde z_1 - \rho\frac{\sigma_2}{\sigma_1}\tilde z_2, \tilde z_2) \triangleq \tilde w^-(\tilde z_1, \tilde z_2), \tag{5.28}$$

where we have used the change of variables (5.8). The reader can verify that if M were equal zero, then in the NT^- region the formula for $\tilde w^-(z_1, z_2)$ agrees with the power series expansion (3.53), except for the $\gamma_3\lambda$ term. The term $-M\lambda$ in the definition of $\tilde w^-$ will be used to create a subsolution.

Differentiating (5.28), we obtain:

$$\begin{cases} w_1^- = \tilde w_1^-, \\ w_2^- = \tilde w_2^- + \rho\frac{\sigma_2}{\sigma_1}\tilde w_1^-, \\ w_{11}^- = \tilde w_{11}^-, \\ w_{12}^- = \tilde w_{12}^- + \rho\frac{\sigma_2}{\sigma_1}\tilde w_{11}^-, \\ w_{22}^- = \tilde w_{22}^- + 2\rho\frac{\sigma_2}{\sigma_1}\tilde w_{12}^- + \left(\rho\frac{\sigma_2}{\sigma_1}\right)^2 \tilde w_{11}^-. \end{cases} \tag{5.29}$$

Remark 5.13. Let (z_1, z_2) be on the western boundary of the NT^-. By definition we have

$$\tilde f_W^-(z_1 + \rho\frac{\sigma_2}{\sigma_1}z_2 - \tilde\theta_1, z_2 - \tilde\theta_2) = 0,$$

for

$$z_1 = \tilde\delta_{W,1}^-(z_2 - \tilde\theta_2) + \tilde\theta_1 - \rho\frac{\sigma_2}{\sigma_1}z_2, \quad z_{2,SW}^- \le z_2 \le z_{2,NW}^-. \tag{5.30}$$

From Lemma 4.8 Assumption B.1.(i) is satisfied. Similar conclusion holds for the eastern boundary.

The northern boundary is simpler. Since the boundary is a straight (radial) line, we have

$$z_2 = \frac{z_{2,NE}^-}{z_{1,NE}^-}z_1, \quad z_{1,NW}^- \le z_1 \le z_{1,NE}^-. \tag{5.31}$$

The southern boundary is treated similarly. For both boundaries, Assumption B.1.(i) is satisfied.

Also, similar to Remark 4.17, for (z_1, z_2) on the western boundary of the NT^- region, the equation

$$\tilde f_W^-(z_1 + \rho\frac{\sigma_2}{\sigma_1}z_2 - \tilde\theta_1, z_2 - \tilde\theta_2) = 0$$

is equivalent to

$$\lambda_1 p\tilde{w}^-(z_1+\rho\frac{\sigma_2}{\sigma_1}z_2,z_2)-(1+\lambda_1(z_1+\rho\frac{\sigma_2}{\sigma_1}z_2))\tilde{w}_1^-(z_1+\rho\frac{\sigma_2}{\sigma_1}z_2,z_2)-\lambda_1 z_2\tilde{w}_2^-(z_1+\rho\frac{\sigma_2}{\sigma_1}z_2,z_2)=0.$$

From (5.29) we have that

$$\lambda_1 p\tilde{w}^- - (1+\tilde{\lambda}_1\tilde{z}_1)\tilde{w}_1^- - \lambda_1\tilde{z}_2\tilde{w}_2^- = \lambda_1 p w^- - (1+\lambda_1 z_1)w_1^- - \lambda_1 z_2 w_2^-,$$

so we also have that $(\mathcal{B}_1(w^-))(z_1,z_2)=0$ there. By a similar argument, we have

$(\mathcal{S}_1(w^-))(z_1,z_2)=0$ on the eastern boundary.

Regarding the southern boundary, we have that

$$\tilde{f}_S^-(\tilde{z}_{1,SW}^- - \tilde{\theta}_1, \tilde{z}_{2,SW}^- - \tilde{\theta}_2)=0,$$

which evaluates to

$$\lambda_2 p\tilde{w}^-(\tilde{z}_{1,SW}^-,\tilde{z}_{2,SW}^-)-(1+\lambda_2\tilde{z}_{2,SW}^-)\tilde{w}_2^-(\tilde{z}_{1,SW}^-,\tilde{z}_{2,SW}^-)-\lambda_2\tilde{z}_{1,SW}^- w_1^-(\tilde{z}_{1,SW}^-,\tilde{z}_{2,SW}^-)$$
$$-\rho\frac{\sigma_2}{\sigma_1}\tilde{w}_1^-(\tilde{z}_{1,SW}^-,\tilde{z}_{2,SW}^-)=0.$$

From (5.29) we have that

$$\lambda_2 p\tilde{w}^- - (1+\tilde{\lambda}_2\tilde{z}_2)\tilde{w}_2^- - \lambda_2\tilde{z}_1\tilde{w}_1^- - \rho\frac{\sigma_2}{\sigma_1}\tilde{w}_1^- = \lambda_2 p w^- - (1+\lambda_2 z_2)w_2^- - \lambda_2 z_1 w_1^-.$$

Thus we also have that $(\mathcal{B}_2(w^-))(z_{1,SW}^-,z_{2,SW}^-)=0$. By a similar argument, we have

$(\mathcal{S}_2(w^-))(z_{1,NE}^-,z_{2,NE}^-)=0.$ □

We extend $w^- \in C^2(\overline{NT^-})$ to the rest of the solvency region using Proposition B.11, and from now on, we will refer to w^- as the extended function. Assumptions B.1 and the requirements of Lemma B.15 are satisfied in the eastern and western regions, as shown in Remark 5.13, so we can apply Lemma B.15, which asserts that w^- will be continuously differentiable across the eastern and western boundaries of the NT^- region, excluding the corners. Moreover, in the northeastern and southwestern corners we can also apply Lemma B.17 to get that w^- is continuously differentiable there too. We caution the reader that w^- is not a continuously differentiable function, as we cannot apply Theorem B.2 as before. We only have continuous differentiability across the eastern and western boundaries, including only the northeastern and southwestern corners, out of the total four. In any case, The extension of w^- has a zero boundary condition on the boundary of the solvency region.

5.3 Verification that (w^-, NT^-) is a subsolution.

It suffices to verify

$$\mathcal{D}w^-(z_1,z_2)-\widetilde{U}\left(pw^-(z_1,z_2)-(z_1 w_1^-(z_1,z_2)+z_2 w_2^-(z_1,z_2))\right)\leq 0 \quad (z_1,z_2)\in\overline{NT^-}. \quad (5.32)$$

We use the facts that $z_i - \theta_i = O\big(\lambda^{1/3}\big)$, so $h_i(z_i - \theta_i) = O\big(\lambda^{4/3}\big)$, $h_i'(z_i - \theta_i) = O(\lambda)$.

From (5.8) and (5.29) it follows that

$$
\begin{aligned}
& z_1 w_1^-(z_1, z_2) + z_2 w_2^-(z_1, z_2) = \tilde{z}_1 \tilde{w}_1^-(\tilde{z}_1, \tilde{z}_2) + \tilde{z}_2 \tilde{w}_2^-(\tilde{z}_1, \tilde{z}_2), \\
& z_1^2 w_{11}^-(z_1, z_2) + 2 z_1 z_2 w_{12}^-(z_1, z_2) + z_2^2 w_{22}^-(z_1, z_2) \\
& = \tilde{z}_1^2 \tilde{w}_{11}^-(\tilde{z}_1, \tilde{z}_2) + 2 \tilde{z}_1 \tilde{z}_2 \tilde{w}_{12}^-(\tilde{z}_1, \tilde{z}_2) + \tilde{z}_2^2 \tilde{w}_{22}^-(\tilde{z}_1, \tilde{z}_2)
\end{aligned}
$$

Also because

$$
\begin{aligned}
\sigma_1^2 z_1^2 + \sigma_2^2 z_2^2 + 2\rho\sigma_1\sigma_2 z_1 z_2 &= \tilde{\sigma}_1^2 \tilde{z}_1^2 + \tilde{\sigma}_2^2 \tilde{z}_2^2, \\
\sigma_1^2 (z_1 - \theta_1)^2 + \sigma_2^2 (z_2 - \theta_2)^2 + 2\rho\sigma_1\sigma_2 (z_1 - \theta_1)(z_2 - \theta_2) &= \tilde{\sigma}_1^2 (\tilde{z}_1 - \tilde{\theta}_1)^2 + \tilde{\sigma}_2^2 (\tilde{z}_2 - \tilde{\theta}_2)^2, \\
\mu_1 z_1 + \mu_2 z_2 &= \tilde{\mu}_1 \tilde{z}_1 + \tilde{\mu}_2 \tilde{z}_2,
\end{aligned}
$$

we have

$$
\begin{aligned}
& \mathcal{D} w^-(z_1, z_2) - \widetilde{U}(p w^-(z_1, z_2) - (z_1 w_1^-(z_1, z_2) + z_2 w_2^-(z_1, z_2))) \\
&= \left[(1-p)A + \frac{p(1-p)}{2} \left[\sigma_1^2 (z_1 - \theta_1)^2 + \sigma_2^2 (z_2 - \theta_2)^2 + 2\rho\sigma_1\sigma_2 (z_1 - \theta_1)(z_2 - \theta_2) \right] \right] w^- \\
&\quad + \left[z_1 w_1^- + z_2 w_2^- \right] \left[r + \sum_{i=1}^2 \mu_i z_i - (1-p)(\sigma_1^2 z_1^2 + \sigma_2^2 z_2^2 + 2\rho\sigma_1\sigma_2 z_1 z_2) \right] \\
&\quad - \frac{1}{2} \left[z_1^2 w_{11}^- + 2 z_1 z_2 w_{12}^- + z_2^2 w_{22}^- \right] \left[\sigma_1^2 z_1^2 + \sigma_2^2 z_2^2 + 2\rho\sigma_1\sigma_2 z_1 z_2 \right] \\
&\quad - \widetilde{U} \left(p w^- - z_1 w_1^- - z_2 w_2^- \right) \\
&= \left[(1-p)A + \frac{p(1-p)}{2} \left[\tilde{\sigma}_1^2 (\tilde{z}_1 - \tilde{\theta}_1)^2 + \tilde{\sigma}_2^2 (\tilde{z}_2 - \tilde{\theta}_2)^2 \right] \right] \tilde{w}^- \\
&\quad + \left[\tilde{z}_1 \tilde{w}_1^- + \tilde{z}_2 \tilde{w}_2^- \right] \left[r + \sum_{i=1}^2 \tilde{\mu}_i \tilde{z}_i - (1-p)(\tilde{\sigma}_1^2 \tilde{z}_1^2 + \tilde{\sigma}_2^2 \tilde{z}_2^2) \right] \\
&\quad - \frac{1}{2} \left[\tilde{z}_1^2 \tilde{w}_{11}^- + 2 \tilde{z}_1 \tilde{z}_2 \tilde{w}_{12}^- + \tilde{z}_2^2 \tilde{w}_{22}^- \right] \left[\tilde{\sigma}_1^2 \tilde{z}_1^2 + \tilde{\sigma}_2^2 \tilde{z}_2^2 \right] - \widetilde{U} \left(p \tilde{w}^- - \tilde{z}_1 \tilde{w}_1^- - \tilde{z}_2 \tilde{w}_2^- \right).
\end{aligned}
$$

We conclude that there exists M, independent of λ, such that $(\mathcal{D} - \widetilde{\mathcal{U}})(w^-)(z_1, z_2) \le 0$ by repeating Step 3 from the uncorrelated case.

5.4 w^- is a viscosity subsolution.

We now have all the pieces to conclude that w^- is a subsolution. Let $(z_1, z_2) \in \mathcal{S}_u$, $\varphi \in C^2(\mathcal{S}_u)$ satisfy $\varphi \ge w^-$ and $\varphi(z_1, z_2) = w^-(z_1, z_2)$.

Case 1: $(z_1, z_2) \in NT^-$.

In this case, in section 5.3 we showed that $\left(\mathcal{D} - \widetilde{\mathcal{U}}\right)(w^-) \leq 0$, and similar to Remark 4.6, since $\varphi - w^-$ attains a global minimum at (z_1, z_2), we must have $\nabla\varphi(z_1, z_2) = \nabla w^-(z_1, z_2)$ and $\nabla^2\varphi(z_1, z_2) \geq \nabla^2 w^-(z_1, z_2)$. Thus, we have

$$-\frac{1}{2}(z_1, z_2)\nabla^2\varphi(z_1, z_2)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \leq -\frac{1}{2}(z_1, z_2)\nabla^2 w^-(z_1, z_2)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

so $(\mathcal{D}(\varphi))(z_1, z_2) \leq (\mathcal{D}(w^-))(z_1, z_2)$, thus $\left(\mathcal{D} - \widetilde{\mathcal{U}}\right)(\varphi)(z_1, z_2) \leq \left(\mathcal{D} - \widetilde{\mathcal{U}}\right)(w^-)(z_1, z_2) \leq 0$, and

$$(\mathcal{H}(\varphi))(z_1, z_2) \leq 0.$$

Case 2: (z_1, z_2) is a point in one of the region inside $\mathcal{S}_u \backslash \overline{NT^-}$ or the northeastern or the southwestern corner of the NT^- region

In all those cases, w^- is continuously differentiable at (z_1, z_2). For instance, if (z_1, z_2) is inside either southern or southwestern or southeastern region, then by construction $(\mathcal{B}_2(w^-))(z_1, z_2) = 0$. Since $\varphi - w^-$ attains a global minimum at (z_1, z_2), we must have $\nabla\varphi(z_1, z_2) = \nabla w^-(z_1, z_2)$, thus $(\mathcal{B}_2(\varphi))(z_1, z_2) = 0$, so $(\mathcal{H}(\varphi))(z_1, z_2) \leq 0$.

Case 3: (z_1, z_2) is on the southern or the northern boundary of NT^- excluding the corners

This case is handled similarly to Case 2, once we note that in the operator $\left(\mathcal{D} - \widetilde{\mathcal{U}}\right)(w^-)$ only radial derivatives of w^- appear, and by construction, the northern and southern boundaries of the NT^- region are radial lines. Furthermore, w^- is defined there by (5.27) and (5.28) and is therefore continuously twice differentiable in the radial direction on these boundaries, excluding the corners.

Case 4: (z_1, z_2) on one of the boundary between the northern, southern, eastern and western regions, excluding corners of the NT^- region

Even though on some of these boundaries (e.g. the boundary between the eastern and the southeastern regions) we have that w^- is continuously differentiable, by Theorem B.2, the following argument is applicable for all the cases, even when Theorem B.2 cannot be applied (e.g. the boundary between the southern and the southeastern regions). Indeed, consider the boundary between southern and the southeastern regions, where w^- is not continuously differentiable, but it is continuously differentiable along that boundary in the direction of the boundary, which is a straight line connecting the corner $(z^-_{1,SE}, z^-_{2,SE})$ with $(0, -\frac{1}{\lambda_2})$. The direction of this line is $\begin{pmatrix} \lambda_2 z^-_{1,SE} \\ 1 + \lambda_2 z^-_{2,SE} \end{pmatrix}$. It follows that if there exists $\varphi \in C^2(\mathcal{S}_u)$ satisfy $\varphi \geq w^-$ and $\varphi(z_1, z_2) = w^-(z_1, z_2)$, then arguing as before

$$\begin{pmatrix} \lambda_2 z^-_{1,SE} \\ 1 + \lambda_2 z^-_{2,SE} \end{pmatrix} \nabla\varphi(z_1, z_2) = \begin{pmatrix} \lambda_2 z^-_{1,SE} \\ 1 + \lambda_2 z^-_{2,SE} \end{pmatrix} \nabla w^-(z_1, z_2)$$

Thus $(\mathcal{B}_2(\varphi))(z_1, z_2) = (\mathcal{B}_2(w^-))(z_1, z_2) = 0$. The other boundaries are treated similarly.

Case 5: (z_1, z_2) is the northwestern or the southeastern corner

This case, is treated similarly to Case 4. Consider, for instance $(z_1, z_2) = (z^-_{1,NW}, z^-_{2,NW})$. By Remark B.8, w^- has a continuous directional derivative, in the direction of the characteristic line of $(\mathcal{B}_1(w^-)) = 0$ at $(z^-_{1,NW}, z^-_{2,NW})$. Then, similar to Case 4, it follows that if there exists $\varphi \in C^2(\mathcal{S}_u)$ satisfy $\varphi \geq w^-$ and $\varphi(z_1, z_2) = w^-(z_1, z_2)$, then arguing as before

$$\begin{pmatrix} 1 + \lambda_1 z^-_{1,NW} \\ \lambda_1 z^-_{2,NW} \end{pmatrix} \nabla \varphi(z^-_{1,NW}, z^-_{2,NW}) = \begin{pmatrix} 1 + \lambda_1 z^-_{1,NW} \\ \lambda_1 z^-_{2,NW} \end{pmatrix} \nabla w^-(z^-_{1,NW}, z^-_{2,NW})$$

Thus

$$(\mathcal{B}_1(\varphi))(z^-_{1,NW}, z^-_{2,NW}) = \big(\mathcal{B}_1(w^-)\big)(z^-_{1,NW}, z^-_{2,NW}) = 0.$$

This concludes the proof that w^- is a subsolution of (2.26). □

5.5 Conclusion

We note that $w^-(\theta_1, \theta_2) = \frac{A^{p-1}}{p} - \tilde{\gamma}_2 \lambda^{\frac{2}{3}} - M\lambda$, and by Proposition B.11 $w^-|_{\partial \mathcal{S}_u} = 0$, so the Comparison Theorem 4.5 implies

$$\frac{A^{p-1}}{p} - \tilde{\gamma}_2 \lambda^{\frac{2}{3}} - M\lambda \leq u(\theta_1, \theta_2).$$

Let also w^+ be the supersolution for the auxiliary problem, which was constructed in Section "Sub- and Supersolutions for Independent Futures Contracts". It follows that $w^+(\theta_1, \theta_2) = \frac{A^{p-1}}{p} - \tilde{\gamma}_2 \lambda^{\frac{2}{3}} + M\lambda$. Then by Lemma 5.5 we also have that

$$u(\theta_1, \theta_2) \leq \frac{A^{p-1}}{p} - \tilde{\gamma}_2 \lambda^{\frac{2}{3}} + M\lambda.$$

This finishes the proof of the second main Theorem 5.4.

□

Note that Corollary 4.2 and Lemma 4.3 still hold, and we also have that for fixed $(z_1, z_2) \in \mathcal{S}_u$, the value function satisfies

$$u(z_1, z_1) = \frac{1}{p} A^{p-1} - A^{p-2} \lambda^{\frac{2}{3}} \sum_{i=1}^{2} \sqrt[3]{\frac{9}{32} \tilde{\alpha}_i^2 (1-p) \tilde{\sigma}_i^2 \tilde{\theta}_i^4 \left(\tilde{\sigma}_1^2 \tilde{\theta}_1^2 + \tilde{\sigma}_2^2 \tilde{\theta}_2^2 \right)^2} + O(\lambda),$$

and the optimal consumption is

$$c = \left(A + \frac{p}{1-p} \tilde{\gamma}_2 \lambda^{\frac{2}{3}} \right) + O(\lambda).$$

6 Results and Conclusions

In this section we numerically show the impact of correlation on the value function, consumption and the shape of the core region. In all of the following computations we have used $\lambda = 0.001$, $\mu_1 = 0.15$, $\mu_2 = 0.1$, $\sigma_1 = 0.4$, $\sigma_2 = 0.2$, $\alpha_1 = \alpha_2 = 1$, $p = 0.25$, $r = 0.07$ and $\beta = 0.1$.

Figure 3: Changes in $\tilde{\gamma}_2$ as a function of ρ.

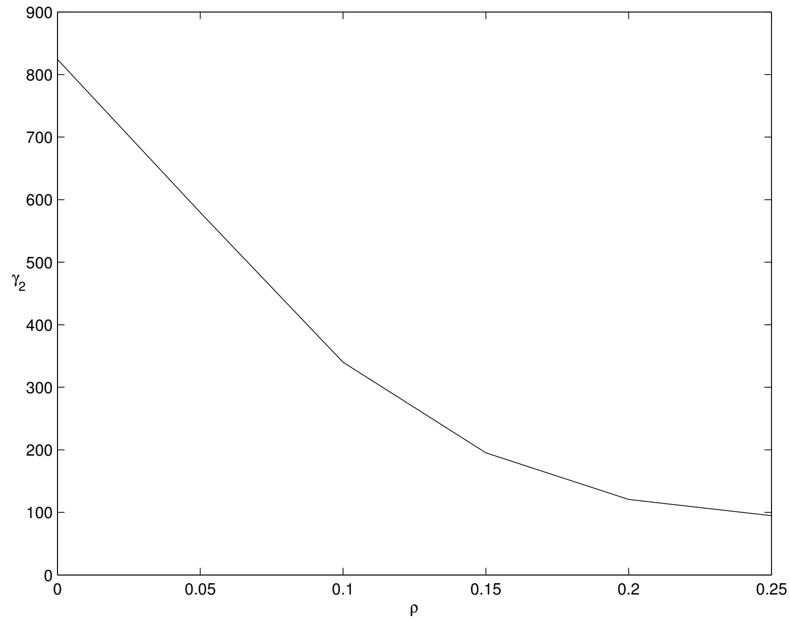
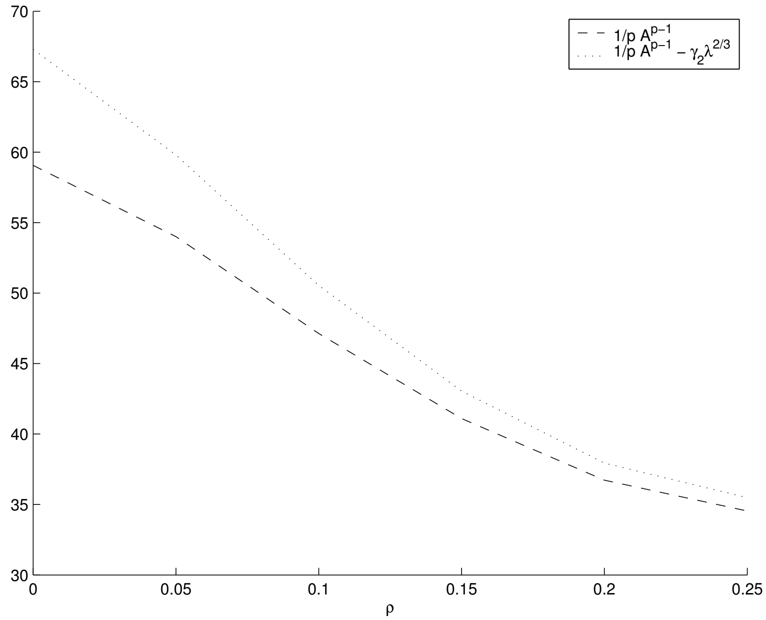


Figure 3 shows the impact of increasing ρ on the $\tilde{\gamma}_2$ coefficient. We observe a decrease in $\tilde{\gamma}_2$ as a function of ρ. In the case of zero transaction cost, the value function u is $\frac{A^{p-1}}{p}$, which is decreasing in ρ. This is not surprising, since the mean return of the optimal portfolio decreases and its variance increases as ρ increases. However, since $\tilde{\gamma}_2$ decreases as ρ increases, it cause the value function u in case of positive transaction cost to decrease slower than in case zero transaction cost (Figure 4). We conjecture that it is due to the change in the shape of NT region, which causes less deviation from the Merton optimal ratio that would increase the value function at the $O(\lambda^{\frac{2}{3}})$.

Figure 5 illustrates the impact of increasing ρ on the shape of core region. Figure 6 shows the same core regions as in Figure 5, but centers them at $(0, 0)$. While the core region is not the optimal NT region, we believe that it is close to it. Specifically, we conjecture that the location of the SW and the NE corners of the optimal NT region match the location of these corners of the core region up to $O(\lambda^{\frac{2}{3}})$. Moreover, comparing these figures with Figures 6.8 and 6.9 from Muthuraman & Kumar [43], we see that the changes of the core region as a function of ρ are similar to changes of the optimal NT region. Specifically, we also find the the main diagonal, connecting NE and SW corners, shortens, and the secondary diagonal elongates. The intuitive explanation is that with the increase of ρ, the assets are more correlated, and the investor would be less willing to tolerate a simultaneous increase

Figure 4: Changes in value function $u = \frac{A^{p-1}}{p}$ in case of no transaction costs, and the $O(\lambda)$ approximation to the value function $u = \frac{A^{p-1}}{p} - \tilde{\gamma}_2 \lambda^{\frac{2}{3}}$ as a function of ρ.



or decrease in both assets. Alternatively, the investor would be more tolerant of swings in prices in opposite directions, since with higher correlation an increase in the price of one asset is a better hedge against a drop in the price of the other asset.

Figure 5: Changes of the Core region as a function of ρ.

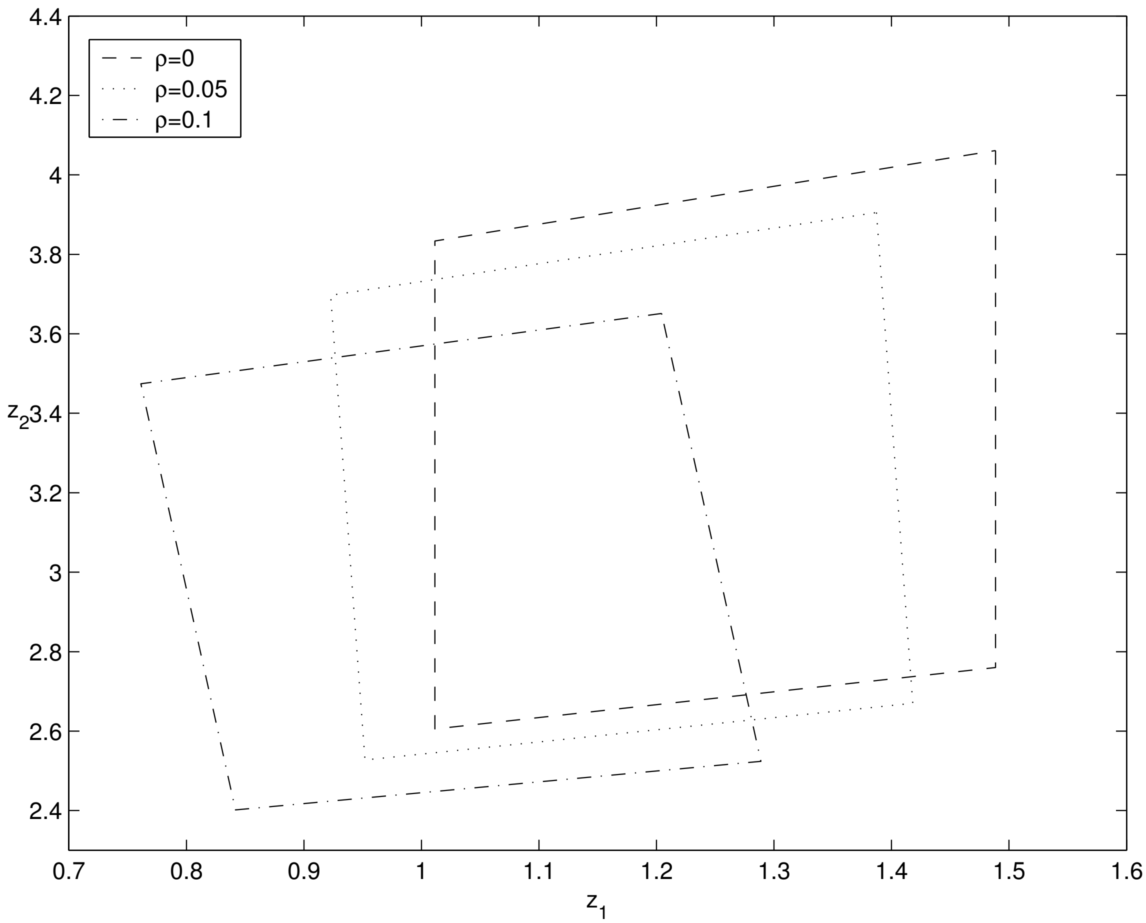
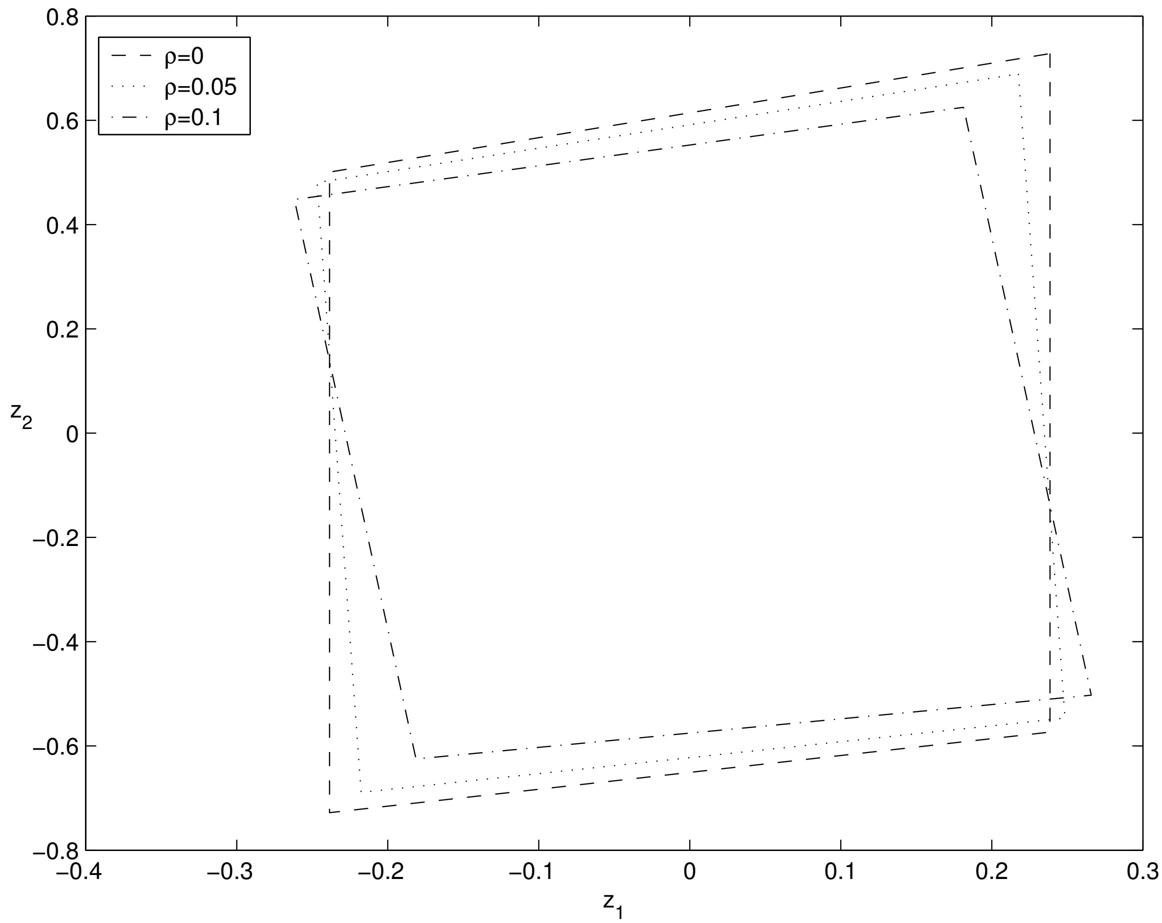


Figure 6: Changes of the Core region as a function of ρ.



A Appendix

Proposition A.1. The value function v defined by (2.11) is concave.

PROOF: Note that an equivalent definition to (2.11) of the value function is

$$v(y_1, y_2, x) = \sup_{C, L_i, M_i} E\left[\int_0^\infty e^{-\beta t} U(C(t))dt\right]. \tag{A.1}$$

Let (y_1^1, y_2^1, x^1) and (y_1^2, y_2^2, x^2) be in $\overline{\mathcal{S}_v}$, and let $\gamma \in (0,1)$, $(L_i^1, M_i^1, C^1)_{i=1,2} \in \mathcal{A}(y_1^1, y_2^1, x^1)$, and $(L_i^2, M_i^2, C^2)_{i=1,2} \in \mathcal{A}(y_1^2, y_2^2, x^2)$ be given. The linearity of (2.3) and (2.2) implies that

$$\begin{aligned}\big(\gamma L_i^1 + (1-\gamma)L_i^2, \gamma M_i^1 + (1-\gamma)M_i^2, \gamma C^1 + (1-\gamma)C^2\big)_{i=1,2}\\ \in \mathcal{A}(\gamma y_1^1 + (1-\gamma)y_1^2, \gamma y_2^1 + (1-\gamma)y_2^2, \gamma x^1 + (1-\gamma)x^2).\end{aligned}$$

Because U is concave, we have

$$\begin{aligned}v\left(\gamma y_1^1 + (1-\gamma)y_1^2, \gamma y_2^1 + (1-\gamma)y_2^2, \gamma x^1 + (1-\gamma)x^2\right)\\ \geq E\int_0^\infty e^{-\beta t} U\big(\gamma C^1(t) + (1-\gamma)C^2(t)\big)dt\\ \geq \gamma\, E\int_0^\infty e^{-\beta t} U\big(C^1(t)\big)dt + (1-\gamma)\, E\int_0^\infty e^{-\beta t} U\big(C^2(t)\big)dt.\end{aligned}$$

Maximizing the right-hand side over $(L_i^1, M_i^1, C^1)_{i=1,2} \in \mathcal{A}(y_1^1, y_2^1, x^1)$ and $(L_i^2, M_i^2, C^2)_{i=1,2}$ $\in \mathcal{A}(y_1^2, y_2^2, x^2)$, we obtain

$$v\left(\gamma y_1^1 + (1-\gamma)y_1^2, \gamma y_2^1 + (1-\gamma)y_2^2, \gamma x^1 + (1-\gamma)x^2\right) \geq \gamma v(y_1^1, y_2^1, x^1) + (1-\gamma)v(y_1^2, y_2^2, x^2).$$

□

Corollary A.2. The value function v is continuous on $\mathcal{S}_v$.

PROOF: A concave function is continuous on the interior of its domain ([44], Theorem 10.1).

The following Propositions A.3, A.4 and A.7, Lemma A.6 and Corollary A.9 are all adaptations from [46] and their proofs are provided here for completeness. The goal, which we achieve in Corollary A.9, is to prove that v is continuous on $\partial\mathcal{S}_v$.

Proposition A.3. Define

$$C_* = \frac{\beta - rp}{1-p}.$$

Assume $0 < p < 1$ and $A > 0$, then $C_* > 0$ and $\forall (y_1, y_2, x) \in \overline{\mathcal{S}_v}$ the value function has the lower bound

$$v(y_1, y_2, x) \geq \frac{1}{p} C_*^{p-1}\big(x - \lambda_1|y_1| - \lambda_2|y_2|\big)^p \tag{A.2}$$

On $\partial\mathcal{S}_v$, v coincides with these lower bounds,which are 0.

Proof: Remark 2.1 gives us the claimed values of v on $\partial\mathcal{S}_v$. For $(y_1, y_2, x) \in \mathcal{S}_v$, let γ be a constant satisfying $\gamma > \max\{0, -(\beta - rp)/p\}$ and consider an admissible policy by liquidating the initial position in futures contracts and investing only in the money market account. Then $X(0) = x - \lambda_1|y_1| - \lambda_2|y_2|$, $Y_i(0) = 0$, $i = 1, 2$ and with proportional consumption $c(t) = \gamma$. Then $X(0) > 0$ and $X(t) = X(0)e^{(r-\gamma)t}$. It follows that the value function is greater then the integral of the discounted utility of the consumption of this strategy $v(y_1, y_2, x) \geq X^p(0)\gamma^p/[p(\beta - rp + \gamma p)]$. If $(\beta - rp)/p \leq 0$, we can let $\gamma \downarrow -(\beta - rp)/p$ and this lower bound on v converges to ∞ for all $(y_1, y_2, x) \in \mathcal{S}_v$. This contradicts the fact that $v(y_1, y_2, x) \leq \frac{A^{p-1}}{p}x^p$, the right hand side being the Merton value function with zero transaction cost. Thus, $C_* > 0$. Maximizing $X^p(0)\gamma^p/[p(\beta - rp + \gamma p)]$ over γ we get the desired lower bound.

We will now concentrate on the case $\rho \geq 0$ and establish the continuity of v on the boundary of $\mathcal{S}_v$, where $x - \lambda_1 y_1 - \lambda_2 y_2 = 0$. The proof for $\rho < 0$ and for other boundaries is analogous.

Consider a function $\varphi : \overline{\mathcal{S}_v} \to [0, \infty)$ of the form

$$\varphi(y_1, y_2, x) = \frac{1}{p}\overline{A}^{p-1}(x - \lambda_1 y_1 - \lambda_2 y_2)^p \qquad \forall (y_1, y_2, x) \in \overline{\mathcal{S}_v}, \tag{A.3}$$

where $0 < p < 1$ and $\overline{A} > 0$ is a positive constant. Since $(y_1, y_2, x) \in \overline{\mathcal{S}_v}$ we have that $x - \lambda_1 y_1 - \lambda_2 y_2 \geq 0$, so φ is well defined. Direct computation reveals that for $i = 1, 2$ we have

$$\lambda_i \varphi_x \pm \varphi_i = \overline{A}^{p-1}(x - \lambda_1 y_1 - \lambda_2 y_2)^{p-1}\big(\lambda_i \mp \lambda_i\big) \geq 0 \qquad \text{on } \overline{\mathcal{S}_v}, \tag{A.4}$$

and

$$\begin{aligned}
&(\mathcal{L}\varphi)(y_1, y_2, x) - \widetilde{U}(\varphi_x(y_1, y_2, x)) = \overline{A}^{p-1}(x - \lambda_1 y_1 - \lambda_2 y_2)^p \qquad\qquad (A.5)\\
&\qquad \times\Big[\frac{\beta}{p} - \frac{rx + \mu_1 y_1 + \mu_2 y_2}{x - \lambda_1 y_1 - \lambda_2 y_2} + \frac{1}{2}(1-p)\frac{\sigma_1^2 y_1^2 + \sigma_2^2 y_2^2 + 2\rho\sigma_1\sigma_2 y_1 y_2}{(x - \lambda_1 y_1 - \lambda_2 y_2)^2} - \frac{1-p}{p}\overline{A}\Big]\\
&= \overline{A}^{p-1}(x - \lambda_1 y_1 - \lambda_2 y_2)^p\\
&\qquad \times\Big[\frac{\beta - rp}{p} - \frac{\sum_{i=1}^2(\mu_i + \lambda_i r)y_i}{x - \lambda_1 y_1 - \lambda_2 y_2} + \frac{1}{2}(1-p)\frac{\sigma_1^2 y_1^2 + \sigma_2^2 y_2^2 + 2\rho\sigma_1\sigma_2 y_1 y_2}{(x - \lambda_1 y_1 - \lambda_2 y_2)^2} - \frac{1-p}{p}\overline{A}\Big]\\
&= \overline{A}^{p-1}(x - \lambda_1 y_1 - \lambda_2 y_2)^p\Big[\rho(1-p)\frac{1}{2}\left(\frac{\sigma_1 y_1}{x - \lambda_1 y_1 - \lambda_2 y_2} + \frac{\sigma_1 y_1}{x - \lambda_1 y_1 - \lambda_2 y_2}\right)^2\\
&\qquad +(1-\rho)(1-p)\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\sigma_i y_i}{x - \lambda_1 y_1 - \lambda_2 y_2} - \frac{\mu_i + \lambda_i r}{\sigma_i(1-\rho)(1-p)}\right)^2\\
&\qquad +\frac{\beta - rp}{p} - \sum_{i=1}^{2}\frac{(\mu_i + \lambda_i r)^2}{2\sigma_i^2(1-\rho)(1-p)} - \frac{1-p}{p}\overline{A}\Big], \quad \forall (y_1, y_2, x) \in \overline{Sv}.
\end{aligned}$$

Regarding p as a variable, consider the equation

$$B(p) \triangleq \frac{\beta - rp}{p} - \sum_{i=1}^{2}\frac{(\mu_i + \lambda_i r)^2}{2\sigma_i^2(1-\rho)(1-p)} = 0, \tag{A.6}$$

which has a unique solution $\overline{p} \in (0,1)$. Define

$$\overline{A}(p) = \frac{\beta - rp}{1-p} - \sum_{i=1}^{2} \frac{p(\mu_i + \lambda_i r)^2}{2\sigma_i^2(1-\rho)(1-p)^2} = \frac{p}{1-p} B(p); \tag{A.7}$$

if $0 < p < \overline{p}$, then $\overline{A}(p) > 0$, and replacing $\overline{A}$ by $\overline{A}(p)$ in (A.3), we have

$$\mathcal{L}\varphi - \widetilde{U}(\varphi_x) \geq 0 \qquad \text{on } \mathcal{S}_v. \tag{A.8}$$

Proposition A.4. Assume $\rho \geq 0,\ 0 < p < \overline{p}$ and A from (2.14) is strictly positive. With $\overline{A}(p)$ defined by (A.7), we have

$$v(y_1, y_2, x) \leq \frac{1}{p} \overline{A}^{p-1}(p)(x - \lambda_1 y_1 - \lambda_2 y_2)^p \qquad \forall (y_1, y_2, x) \in \overline{\mathcal{S}_v}.$$

Proof: Let φ be given by (A.3) with $\overline{A} = \overline{A}(p)$. Let $(y_1, y_2, x) \in \mathcal{S}_v$ and $(L_i, M_i, C)_{i=1,2} \in \mathcal{A}(y_1, y_2, x)$ be given. Inequality (A.4) shows that φ is non increasing in the direction of jumps of the corresponding state process, i.e.,

$$\varphi(Y_1(s), Y_2(s), X(s)) \leq \varphi(Y_1(s-), Y_2(s-), X(s-)) \qquad \forall s \geq 0. \tag{A.9}$$

Choose an increasing sequence $\{K_n\}_{n=1}^{\infty}$ of compact subsets of $\mathcal{S}_v$ containing (y_1, y_2, x) and whose union is $\mathcal{S}_v$, and define $\tau_n \triangleq n \wedge \inf \left\{ t \geq 0; (Y_1(t), Y_2(t), X(t)) \notin K_n \right\}$. Then

$$E\left[\int_0^{\tau_n} e^{-\beta s} Y_i(s) \varphi_x(Y_1(s), Y_2(s), X(s)) dB_i(s) \right] = 0, \quad i = 1, 2$$

for each n. From Itô's rule, (A.4), (A.8), (A.9), (2.17) and the non negativity of φ, we have

$$\varphi(y_1, y_2, x) \geq E \int_0^{\tau_n} e^{-\beta s} U(C(s)) ds.$$

Let $n \to \infty$ and then maximize the right-hand side over $(L_i, M_i, C)_{i=1,2}$ to obtain the desired result.

Remark A.5. An alternative proof for Proposition A.4 would be to note that φ is a viscosity supersolution to the HJB equation (2.20) with boundary condition $\varphi\Big|_{\partial \mathcal{S}_v} \geq 0$. By Theorem 2.4 the value function v is a viscosity solution to the HJB equation (2.20). We now can appeal to Comparison Theorem 4.5 to conclude that $\varphi \geq v$ on $\overline{\mathcal{S}_v}$.

Finally, we examine the case $\overline{p} \leq p < 1$, which is not covered by Proposition A.4. The following proof is more complicated then that of Proposition A.4, but it covers all the cases $0 < p < 1$. To do that, we need a brief digression on the manner in which the state process can approach $\partial \mathcal{S}_v$.

Lemma A.6. For $n = 1, 2, \dots,$ define

$$F_n = \Big\{ (y_1, y_2, x) \in \mathcal{S}_v ; x - \lambda_1 |y_1| - \lambda_2 |y_2| \geq \frac{1}{n} \Big\}. \tag{A.10}$$

For $(y_1, y_2, x) \in \mathcal{S}_v$ and $(L_i, M_i, C)_{i=1,2} \in \mathcal{A}(y_1, y_2, x)$, let (Y_1, Y_2, X) be given by (2.2)–(2.4) and define

$$\nu_n \triangleq \inf \big\{ t \geq 0; \big(Y_1(t), Y_2(t), X(t)\big) \notin F_n \big\}, \tag{A.11}$$

$$\nu \triangleq \inf \big\{ t \geq 0; \big(Y_1(t), Y_2(t), X(t)\big) = (0, 0, 0) \big\}, \tag{A.12}$$

Then $\nu_n \uparrow \nu$ almost surely as $n \to \infty$.

Proof: Define $\nu_\infty = \lim_{n\to\infty} \nu_n$. Then clearly $\nu_\infty \leq \nu$, and we have only to prove the reverse inequality. Suppose $\nu_\infty < \infty$. Then $(Y_1(\nu_\infty-), Y_2(\nu_\infty-), X(\nu_\infty-)) \in \partial\mathcal{S}_v$. The argument in Remark 2.1 shows that $(Y_1(\nu_\infty), Y_2(\nu_\infty), X(\nu_\infty)) = (0, 0, 0)$. Therefore, $\nu \leq \nu_\infty$.

Proposition A.7. Assume $\rho \geq 0$, $0 < p < 1$ and A from (2.14) is strictly positive. Choose $0 < \varepsilon < \lambda_1 \wedge \lambda_2$, and such that $\varepsilon < \frac{1}{\sum_{i=1}^2 \frac{4(\mu_i + \lambda_i r)}{\min\{\sigma_1^2, \sigma_2^2\}(1-\rho)(1-p)}}$, and define

$$\overline{A} \triangleq \min \left\{ \Bigg[\sup_{\substack{(x, y_1, y_2) \in \partial D \setminus (0,0,0), \\ x = \sum_{i=1}^2 (\lambda_i + \varepsilon) y_i}} \frac{p v(y_1, y_2, x)}{(\varepsilon(y_1 + y_2))^p} \Bigg]^{1/(p-1)}, C_* \right\}, \tag{A.13}$$

where

$$D = \Big\{ (y_1, y_2, x) \in \mathcal{S}_v ; \lambda_1 y_1 + \lambda_2 y_2 < x < (\lambda_1 + \varepsilon) y_1 + (\lambda_2 + \varepsilon) y_2 \Big\}. \tag{A.14}$$

Then $0 < \overline{A} \leq C_*$, and for $(y_1, y_2, x) \in \overline{D}$ we have the upper bounds

$$v(y_1, y_2, x) \leq \frac{1}{p} \overline{A}^{p-1} \big(x - \lambda_1 y_1 - \lambda_2 y_2\big)^p. \tag{A.15}$$

Proof: We have that $\overline{A}^{p-1} \geq C_*^{p-1}$, to conclude that $0 < \overline{A}$ we need to show that the supremum in (A.13) is finite. For $(y_1, y_2, x) \in \partial D \setminus (0, 0, 0)$, such that $x = \sum_{i=1}^2 (\lambda_i + \varepsilon) y_i$, it follows that $\varepsilon(y_1+y_2)>0$. We have

$$\begin{aligned}
&\sup_{\substack{(x,y_1,y_2)\in\partial D\setminus(0,0,0),\\ x=\sum_{i=1}^2(\lambda_i+\varepsilon)y_i}} \frac{pv(y_1,y_2,x)}{(\varepsilon(y_1+y_2))^p}\\
=&\sup_{\substack{(x,y_1,y_2)\in\partial D\setminus(0,0,0),\\ x=\sum_{i=1}^2(\lambda_i+\varepsilon)y_i}} pv\left(\frac{y_1}{\varepsilon(y_1+y_2)},\frac{y_2}{\varepsilon(y_1+y_2)},\frac{x}{\varepsilon(y_1+y_2)}\right)\\
\leq&\sup_{\substack{(x,y_1,y_2)\in\partial D\setminus(0,0,0),\\ x=\sum_{i=1}^2(\lambda_i+\varepsilon)y_i}} A^{p-1}\left(\frac{x}{\varepsilon(y_1+y_2)}\right)^p\\
\leq&\ A^{p-1}\left(\max\left\{\frac{\lambda_1+\lambda_2+\varepsilon}{\varepsilon},\frac{(\lambda_1+\lambda_2)(\varepsilon(\lambda_1+\lambda_2)+2\lambda_1\lambda_2)}{2\varepsilon\lambda_1\lambda_2}\right\}\right)^p,
\end{aligned}\tag{A.16}$$

where the upper bound in the first inequality is the value function in case of zero transaction costs. The the last inequality is treated in cases. If $y_1,y_2\geq 0$, then $x=\sum_{i=1}^2(\lambda_i+\varepsilon)y_i\leq(\lambda_1+\lambda_2+\varepsilon)(y_1+y_2)$, and $\frac{x}{\varepsilon(y_1+y_2)}\leq\frac{\lambda_1+\lambda_2+\varepsilon}{\varepsilon}$. If $y_1>0>y_2$, then because $(y_1,y_2,x)\in\partial\,\mathrm{D}\subset\overline{\mathcal{S}_v}$ we have that $x-\lambda_1y_1+\lambda_2y_2\geq 0$, and also $x=\sum_{i=1}^2(\lambda_i+\varepsilon)y_i$. It follows that $\frac{x}{\lambda_1+\epsilon}\leq y_1$ and $y_2\geq\frac{-\varepsilon x}{\varepsilon(\lambda_1+\lambda_2)+2\lambda_1\lambda_2}$. It follows from the choice of ε that $y_1+y_2\geq\frac{2\lambda_1\lambda_2x}{(\lambda_1+\lambda_2)(\varepsilon(\lambda_1+\lambda_2)+2\lambda_1\lambda_2)}$, and we conclude that $\frac{x}{\varepsilon(y_1+y_2)}\leq\frac{(\lambda_1+\lambda_2)(\varepsilon(\lambda_1+\lambda_2)+2\lambda_1\lambda_2)}{2\varepsilon\lambda_1\lambda_2}$. The case when $y_2>0>y_1$ is treated similarly. The case when $y_1,y_2<0$ is impossible, and we have excluded the case when $(y_1,y_2,x)=(0,0,0)$. This proves (A.16).

We conclude that $0<\overline{A}$ and $\overline{A}\leq C_*$, where $C_*=(\beta-rp)/(1-p)>0$ because of Proposition A.3.

Define φ on $\overline{D}$ by

$$\varphi(y_1,y_2,x)\triangleq\frac{1}{p}\overline{A}^{p-1}\big(x-\lambda_1y_1-\lambda_2y_2\big)^p\qquad\forall(y_1,y_2,x)\in\overline{D}.\tag{A.17}$$

Note that $\varphi\geq v$ on ∂D. Indeed, it clearly holds on $\partial D\cap\partial\mathcal{S}_v$, where φ is non-negative and v is zero. Otherwise, for $(y_1,y_2,x)\in\partial D\setminus\partial\mathcal{S}_v$, we have $x-\lambda_1y_1-\lambda_2y_2=\varepsilon(y_1+y_2)$. Then by definition of $\overline{A}$ we have that

$$\varphi(y_1,y_2,x)=\frac{\overline{A}^{p-1}}{p}\big(x-\lambda_1y_1-\lambda_2y_2\big)^p\geq\frac{v(y_1,y_2,x)}{(\varepsilon(y_1+y_2))^p}\big(x-\lambda_1y_1-\lambda_2y_2\big)^p=v(y_1,y_2,x).$$

Define $\varphi=v$ on $\mathcal{S}_v\setminus\overline{D}$. Just as in (A.4), we have

$$\lambda_i\varphi_x\pm\varphi_i=\overline{A}^{p-1}(x-\lambda_1y_1-\lambda_2y_2)^{p-1}\big(\lambda_i\mp\lambda_i\big)\geq 0,\qquad\text{on } D,\ i=1,2.\tag{A.18}$$

Furthermore, similar to (A.5) for $(y_1, y_2, x) \in D$

$$
\begin{aligned}
&(\mathcal{L}\varphi)(y_1,y_2,x) - \widetilde{U}(\varphi_x(y_1,y_2,x)) \qquad\qquad (A.19)\\
&= \overline{A}^{p-1}(x-\lambda_1 y_1-\lambda_2 y_2)^p\Big[\rho(1-p)\frac{1}{2}\left(\frac{\sigma_1 y_1}{x-\lambda_1 y_1-\lambda_2 y_2}+\frac{\sigma_1 y_1}{x-\lambda_1 y_1-\lambda_2 y_2}\right)^2\\
&\qquad +(1-\rho)(1-p)\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\sigma_i y_i}{x-\lambda_1 y_1-\lambda_2 y_2}-\frac{\mu_i+\lambda_i r}{\sigma_i(1-\rho)(1-p)}\right)^2\\
&\qquad +\frac{\beta-rp}{p}-\sum_{i=1}^{2}\frac{(\mu_i+\lambda_i r)^2}{2\sigma_i^2(1-\rho)(1-p)}-\frac{1-p}{p}\overline{A}\Big]\\
&\geq \overline{A}^{p-1}(x-\lambda_1 y_1-\lambda_2 y_2)^{p-1}(1-\rho)(1-p)\frac{1}{2}\sum_{i=1}^{2}\left(\frac{\sigma_i^2 y_i^2}{x-\lambda_1 y_1-\lambda_2 y_2}-\frac{2(\mu_i+\lambda_i r)y_i}{(1-\rho)(1-p)}\right)\\
&\geq 0,
\end{aligned}
$$

because $A \leq C_*$ and by the choice of ε. Indeed, without loss of generality assume that $y_1 \geq y_2$. Then $y_1 > 0$ and we have

$$
\sum_{i=1}^{2}\frac{\sigma_i^2 y_i^2}{x-\lambda_1 y_1-\lambda_2 y_2}-\frac{2(\mu_i+\lambda_i r)y_i}{(1-\rho)(1-p)} \geq \frac{\sigma_1^2 y_1^2}{x-\lambda_1 y_1-\lambda_2 y_2}-\sum_{i=1}^{2}\frac{2(\mu_i+\lambda_i r)y_1}{(1-\rho)(1-p)} \geq 0, \quad (A.20)
$$

because $0 < x - \lambda_1 y_1 - \lambda_2 y_2 < 2\varepsilon y_1$ or $\frac{y_1}{x-\lambda_1 y_1-\lambda_2 y_2} > \frac{1}{2\varepsilon}$, and (A.20) follows from the choice of ε.

Let $(y_1, y_2, x) \in D$ and $(L_i, M_i, C)_{i=1,2} \in \mathcal{A}(y_1, y_2, x)$ be given, and define F_n, ν_n and ν by (A.10)–(A.12). Define also

$$
\begin{aligned}
H_n &\triangleq \big\{(y_1,y_2,x)\in\overline{D}; y_1,y_2\leq n\big\},\\
\tau_n &\triangleq \inf\big\{t\geq 0; \big(Y_1(t),Y_2(t),X(t)\big)\notin H_n\big\},\\
\tau &\triangleq \inf\big\{t\geq 0; \big(Y_1(t),Y_2(t),X(t)\big)\notin \overline{D}\big\},
\end{aligned}
$$

so that $\lim_{n\to\infty}\tau_n = \tau$ almost surely. We show that

$$
\big\{\tau<\infty\big\} = \bigcup_{n=1}^{\infty}\big\{\nu_n\wedge\tau_n=\tau\leq n\big\}. \qquad (A.21)
$$

It is clear that $\{\tau < \infty\}$ contains the union. For the reverse containment, assume $\tau(\omega) < \infty$ for some ω. Then $\tau(\omega) < \nu(\omega)$, for otherwise, $(Y_1(\cdot,\omega), Y_2(\cdot,\omega), X(\cdot,\omega))$ would reach and stick at the origin before exiting $\overline{D}$ and then $\tau(\omega)$ would be ∞. According to Lemma A.6, we must then have $\tau(\omega) < \nu_n(\omega)$ for sufficiently large n. Choose n so large that $\tau(\omega) \leq n$, $\tau(\omega) < \nu_n(\omega)$, and $\{Y_i(t,\omega); 0 \leq t \leq \tau(\omega)\}$, $i = 1, 2$ does not exceed n. Then $\tau_n(\omega) = \tau(\omega)$, we have $\omega \in \{\nu_n \wedge \tau_n = \tau \leq n\}$, and (A.21) is proved.

Inequalities (A.18) show that (A.9) must hold. Moreover, for $0 \leq s \leq \nu_n \wedge \tau_n$, we have for $i = 1, 2$ that $Y_i(s)\varphi_x(Y_1(s), Y_2(s), X(s))$ is bounded, so

$$
E\left[\int_0^{n\wedge\nu_n\wedge\tau_n} e^{-\beta s}Y_i(s)\varphi_x\big(Y_1(s),Y_2(s),X(s)\big)dB_i(s)\right] = 0,\ i=1,2.
$$

Since φ does not increase in the directions of jumps without loss of generality, we can assume that the diffusion $(Y_1(t), Y_2(t), X(t))$ has no jumps on the boundary of D since From these facts, Itô's rule, (A.18), (A.19) and (2.17), we obtain

$$\begin{aligned}
\varphi(y_1, y_2, x) &\geq E\left[e^{-\beta(n\wedge\nu_n\wedge\tau_n)}\varphi\big(Y_1(n\wedge\nu_n\wedge\tau_n), Y_2(n\wedge\nu_n\wedge\tau_n)X(n\wedge\nu_n\wedge\tau_n)\big)\right] \\
&\quad + E\left[\int_0^{n\wedge\nu_n\wedge\tau_n} e^{-\beta s}U\big(C(s)\big)ds\right] \\
&\geq E\Big[1_{\{v_n\wedge\tau_n=\tau\leq n\}}e^{-\beta\tau}v\big(Y_1(\tau), Y_2(\tau), X(\tau)\big)\Big] \\
&\quad + E\left[\int_0^{n\wedge\nu_n\wedge\tau_n} e^{-\beta s}U\big(C(s)\big)ds\right],
\end{aligned}$$

where the second inequality uses the fact that $\varphi \geq v$ on $\mathcal{S}_v \setminus D$. Letting $n \to \infty$, we can use the monotone convergence theorem to establish

$$\varphi(y_1, y_2, x) \geq E\Big[1_{\{\tau<\infty\}}e^{-\beta\tau}v\big(Y_1(\tau), Y_2(\tau), X(\tau)\big) + \int_0^\tau e^{-\beta s}U\big(C(s)\big)ds\Big].$$

Maximizing the right side over $(L_i, M_i, C)_{i=1,2} \in \mathcal{A}(y_1, y_2, x)$ and by dynamic programming principle, we derive (A.15).

□

Remark A.8. Analogous formulations of Propositions A.4 and A.7 and Lemma A.6 can be established for the three other boundaries of the solvency region $\partial\mathcal{S}_v$ and the case when $\rho < 0$.

Corollary A.9. If $0 < p < 1$ and $A > 0$, then v has limit 0 at $\partial\mathcal{S}_v$, i.e., v is continuous on $\overline{\mathcal{S}_v}$.

Proof: Corollary A.2 and Proposition A.3 show that in order to prove the continuity of v on $\overline{\mathcal{S}_v}$, it suffices to show for every $(y_1^0, y_2^0, x^0) \in \partial\mathcal{S}_v$ that

$$\limsup_{\substack{(y_1,y_2,x)\to(y_1^0,y_2^0,x^0)\\(y_1,y_2,x)\in\mathcal{S}_v}} v(y_1, y_2, x) \leq 0.$$

For $(y_1^0, y_2^0, x^0) \in \partial\mathcal{S}_v \setminus \{(0,0,0)\}$, this follows from Proposition A.7 and Remark A.8. For $(y_1^0, y_2^0, x^0) = (0,0,0)$, it follows from homogeneity property (Lemma 2.5). □

Proposition A.10. Let $(x, y) \in \mathcal{S}_v$ be given, then for all $\delta \geq 0$,

$$v(y_1, y_2, x+\delta) \geq \Big(\frac{x+\delta+\sum_{i=1}^2\lambda_i y_i}{x+\sum_{i=1}^2\lambda_i y_i}\Big)^p v(y_1, y_2, x) \tag{A.22}$$

PROOF: This proof is taken from [46], and is provided here for completeness. Define $\gamma = \frac{\delta}{x+\sum_{i=1}^{2}\lambda_i y_i}$, so that

$$\big(y_1(1+\gamma), y_2(1+\gamma), x+\delta-\gamma\sum_{i=1}^{2}\lambda_i y_i\big) = \Big(\frac{x+\delta+\sum_{i=1}^{2}\lambda_i y_i}{x+\sum_{i=1}^{2}\lambda_i y_i}\Big)(y_1, y_2, x).$$

Since the position $\big(y_1(1+\gamma), y_2(1+\gamma), x+\delta-\gamma\sum_{i=1}^{2}\lambda_i y_i\big)$ is reachable from the initial endowment $(y_1, y_2, x+\delta)$ by a trade, it follows that $v(y_1, y_2, x+\delta) \geq v(y_1(1+\gamma), y_2(1+\gamma), x+\delta-\sum_{i=1}^{2}\gamma y_i\big)$, and the claim of the proposition follows from homotheticity of degree p of v.

$\square$

Corollary A.11. Let $(y_1, y_2, x) \in \mathcal{S}_v$ and a differentiable function $\varphi : \mathcal{S}_v \to R$ satisfying $\varphi(y_1, y_2, x) = v(y_1, y_2, x))$ be given. If $\varphi \geq v$ or $\varphi \leq v$ on $\mathcal{S}_v$, then

$$\varphi_x(y_1, y_2, x) \geq pv(y_1, y_2, x)\Big(x+\sum_{i=1}^{2}\lambda_i y_i\Big)^{-1}$$

PROOF: This proof is taken from [46], and is provided here for completeness. If $\varphi \geq v$, Proposition A.10 implies that

$$\begin{aligned}\varphi_x(y_1, y_2, x) &\geq \overline{\lim_{h\downarrow 0}}\frac{1}{h}\big[v(y_1, y_2, x+h) - v(y_1, y_2, x)\big] \\ &\geq \lim_{h\downarrow 0}\frac{1}{h}\Big[\Big(\frac{x+h+\sum_{i=1}^{2}\lambda_i y_i}{x+\sum_{i=1}^{2}\lambda_i y_i}\Big)^p - 1\Big]v(y_1, y_2, x) \\ &= pv(y_1, y_2, x)\Big(x+\sum_{i=1}^{2}\lambda_i y_i\Big)^{-1}.\end{aligned}$$

Similarly if $\varphi \leq v$

$$\begin{aligned}\varphi_x(y_1, y_2, x) &\geq \overline{\lim_{h\downarrow 0}}\frac{1}{h}\big[v(y_1, y_2, x) - v(y_1, y_2, x-h)\big] \\ &\geq \lim_{h\downarrow 0}\frac{1}{h}\Big[\Big(\frac{x+\sum_{i=1}^{2}\lambda_i y_i}{x-h+\sum_{i=1}^{2}\lambda_i y_i}\Big)^p - 1\Big]v(y_1, y_2, x-h) \\ &= pv(y_1, y_2, x)\Big(x+\sum_{i=1}^{2}\lambda_i y_i\Big)^{-1},\end{aligned}$$

where we have also used continuity of v.

$\square$

Theorem A.12. The value function v defined by (2.11) is a viscosity solution of the HJB equation (2.20) on $\mathcal{S}_v$.

We divide the proof of this theorem into two lemmas.

Lemma A.13. The value function v is a viscosity supersolution of (2.20) on $\mathcal{S}_v$.

PROOF: This proof is taken from [46], and is provided here for completeness. Let $(y_1, y_2, x) \in \mathcal{S}_v$ and $\varphi \in C^2(\mathcal{S}_v)$ be given with

$$\varphi \leq v \quad \text{on } \mathcal{S}_v, \qquad \varphi(y_1, y_2, x) = v(y_1, y_2, x).$$

For $\gamma > 0$ sufficiently small, $(y_1 + \gamma, y_2, x - \lambda_1\gamma) \in \mathcal{S}_v$, it follows that $v(y_1, y_2, x) \geq v(y_1 + \gamma, y_2, x - \lambda_1\gamma)$, and thus

$$\varphi\big(y_1 + \gamma, y_2, x - \lambda_1\gamma\big) - \varphi(y_1, y_2, x) \leq v\big(y_1 + \gamma, y_2, x - \lambda_1\gamma\big) - v(y_1, y_2, x) \leq 0.$$

Divide by γ and let $\gamma \downarrow 0$ to obtain

$$\varphi_1(y_1, y_2, x) - \lambda_1\varphi_x(y_1, y_2, x) \leq 0.$$

Similar arguments show that

$$\lambda_2\varphi_x(y_1, y_2, x) - \varphi_2(y_1, y_2, x), \lambda_1\varphi_x(y_1, y_2, x) + \varphi_1(y_1, y_2, x), \lambda_2\varphi_x(y_1, y_2, x) + \varphi_2(y_1, y_2, x) \geq 0.$$

It remains only to show that

$$(\mathcal{L}\varphi)(y_1, y_2, x) - \widetilde{U}\big(\varphi_x(y_1, y_2, x)\big) \geq 0. \tag{A.23}$$

Let $B_\varepsilon(y_1, y_2, x)$ be an open ball with radius ε, centered at (y_1, y_2, x). Choose $\varepsilon > 0$ so that $\overline{B}_\varepsilon(y_1, y_2, x) \subset \mathcal{S}_v$. For $c > 0$, let $(L_i, M_i, C)_{i=1,2}$ be a policy in $\mathcal{A}(y_1, y_2, x)$ with $L_1(s) = L_2(s) = 0$, $M_1(s) = M_2(s) = 0$ and $C(s) = c$ for $0 \leq s \leq \tau_\varepsilon$, where

$$\tau_\epsilon \triangleq \varepsilon \wedge \inf\big\{t \geq 0; \big(Y_1(y), Y_2(t)X(t),\big) \notin \overline{B}_\varepsilon(y_1, y_2, x)\big\}.$$

Itô's rule implies

$$v(y_1, y_2, x) = \varphi(y_1, y_2, x) = E\left[e^{-\beta\tau_\varepsilon}\varphi\big(Y_1(\tau_\varepsilon), Y_2(\tau_\varepsilon), X(\tau_\varepsilon)\big)\right] + E\left[\int_0^{\tau_\varepsilon} e^{-\beta s}\big(\mathcal{L}\varphi + c\varphi_x\big)ds\right].$$

Using the relationship between v and φ and according to the principle of dynamic programming, we obtain

$$\begin{aligned} v(y_1, y_2, x) &\geq E\Big[\int_0^{\tau_\varepsilon} e^{-\beta s}U(c)ds + e^{-\beta\tau_\varepsilon}v\big(Y_1(\tau_\varepsilon), Y_2(\tau_\varepsilon), X(\tau_\varepsilon)\big)\Big] \\ &\geq E\Big[\int_0^{\tau_\varepsilon} e^{-\beta s}U(c)ds + e^{-\beta\tau_\varepsilon}\varphi\big(Y_1(\tau_\varepsilon), Y_2(\tau_\varepsilon), X(\tau_\varepsilon)\big)\Big] \\ &= v(y_1, y_2, x) - E\left[\int_0^{\tau_\varepsilon} e^{-\beta s}\big[\mathcal{L}\varphi - U(c) + c\varphi_x\big]ds\right]. \end{aligned}$$

It follows that

$$E\left[\int_0^{\tau_\varepsilon} e^{-\beta s}\big[(\mathcal{L}\varphi)(Y_1(s),Y_2(s),X(s)) - U(c) + c\varphi_x(Y_1(s),Y_2(s),X(s))\big]ds\right] \geq 0$$

for all ε sufficiently small. This can happen only if

$$\max_{\overline{B}_\varepsilon(y_1,y_2,x)} \big[(\mathcal{L}\varphi) - U(c) + c\varphi_x\big] \geq 0,$$

and as $\varepsilon \downarrow 0$, we see that

$$(\mathcal{L}\varphi)(y_1,y_2,x) - U(c) + c\varphi_x(y_1,y_2,x) \geq 0.$$

Minimization of this expression over $c > 0$ leads to (A.23). □

Lemma A.14. The value function v is a viscosity subsolution of (2.20) on $\mathcal{S}_v$.

Proof: This proof is taken from [46], and is provided here for completeness. Let $(y_1^0, y_2^0, x^0) \in \mathcal{S}_v$ and $\varphi \in C^2(\mathcal{S}_v)$ be given with $\varphi \geq v$ on $\mathcal{S}_v$ and $\varphi(y_1^0,y_2^0,x^0) = v(y_1^0,y_2^0,x^0)$. We argue by contradiction.

Assume the subsolution inequality

$$\min\big\{\mathcal{L}\varphi - \widetilde{U}(\varphi_x), \lambda_1\varphi_x - \varphi_1, \lambda_2\varphi_x - \varphi_2, \lambda_1\varphi_x + \varphi_1, \lambda_2\varphi_x + \varphi_2\big\} \leq 0 \qquad \text{at } (y_1^0,y_2^0,x^0)$$

fails. Then there are constants $\gamma > 0$, $\varepsilon > 0$ such that the set

$$H \triangleq \big\{(y_1,y_2,x); |x - x^0| \leq \varepsilon, |y_1 - y_1^0| \leq \varepsilon, |y_2 - y_2^0| \leq \varepsilon, \big\}$$

is a subset of $\mathcal{S}_v$, and on H,

$$\mathcal{L}\varphi - \widetilde{U}(\varphi_x) \geq \gamma,\ \lambda_1\varphi_x - \varphi_1 \geq \gamma,\ \lambda_2\varphi_x - \varphi_2 \geq \gamma, \lambda_1\varphi_x + \varphi_1 \geq \gamma,\ \lambda_2\varphi_x + \varphi_2 \geq \gamma. \quad \text{(A.24)}$$

For $(L_i, M_i, C)_{i=1,2} \in \mathcal{A}(y_1^0,y_2^0,x^0)$, define $\tau \triangleq \inf\big\{t \geq 0; \big(Y_1(t),Y_2(t),X(t)\big) \notin H\big\}$. According to the principle of dynamic programming, for each $t \in [0,\infty)$,

$$v(y_1^0,y_2^0,x^0) = \qquad \text{(A.25)}$$
$$\sup_{(L_i,M_i,C)_{i=1,2}\in\mathcal{A}(y_1^0,y_2^0,x^0)} E\Big[\int_0^{t\wedge\tau} e^{-\beta s}U\big(C(s)\big)ds + e^{-\beta(t\wedge\tau)}v\big(Y_1(t\wedge\tau),Y_2(t\wedge\tau),X(t\wedge\tau)\big)\Big].$$

Since the only jumps possible are due to buying or selling one or both types of futures contracts, and this does not increase the value function, it does not reduce the above supremum to restrict it to policies satisfying

$$\big(Y_1(\tau_\varepsilon),Y_2(\tau_\varepsilon),X(\tau_\varepsilon)\big) \in \partial H \qquad \text{on } \{\tau < \infty\}. \qquad \text{(A.26)}$$

Let $(L_i, M_i, C)_{i=1,2}$ be such a policy. From Itô's rule, (A.24) and the fact that $U > 0$ we have

$$\begin{aligned}
\varphi(y_1^0, y_2^0, x^0) \;&\geq\; E\left[e^{-\beta(t\wedge\tau)}\varphi\big(Y_1(t\wedge\tau), Y_2(t\wedge\tau), X(t\wedge\tau)\big)\right] + E\left[\int_0^{t\wedge\tau} e^{-\beta s}\big(\mathcal{L}\varphi - \widetilde{U}(\varphi_x)\big)ds\right] \\
&\quad + E\left[\int_0^{t\wedge\tau} e^{-\beta s}\big(\widetilde{U}(\varphi_x) + C(s)\varphi_x\big)ds\right] \qquad \text{(A.27)} \\
&\quad + \gamma\sum_{i=1}^{2} E\left[\int_0^{t\wedge\tau} e^{-\beta s}\big(dL_i(s) + dM_i(s)\big)\right] \\
&\geq\; E\left[e^{-\beta(t\wedge\tau)}v\big(Y_1(t\wedge\tau), Y_2(t\wedge\tau), X(t\wedge\tau)\big)\right] + E\left[\int_0^{t\wedge\tau} e^{-\beta s}U\big(C(s)\big)ds\right] \\
&\quad + \gamma e^{-\beta t}E\big[(t\wedge\tau)\big] + \gamma e^{-\beta t}\sum_{i=1}^{2} E\big[L_i(t\wedge\tau) + M_i(t\wedge\tau)\big] \\
&\quad + e^{-\beta t}E\left[\int_0^{t\wedge\tau}\big[\widetilde{U}(\varphi_x) + C(s)\varphi_x - U\big(C(s)\big)\big]ds\right] \\
&\geq\; \nu(t) + E\Big[e^{-\beta(t\wedge\tau)}v\big(Y_1(t\wedge\tau), Y_2(t\wedge\tau), X(t\wedge\tau)\big) + \int_0^{t\wedge\tau} e^{-\beta s}U\big(C(s)\big)ds\Big],
\end{aligned}$$

where

$$\begin{aligned}
\nu(t) \;=\; \inf_{\substack{(L_i,M_i,C)_{i=1,2}\in\mathcal{A}(y_1^0,y_2^0,x^0),\\ (A.26)\text{holds.}}} &\Bigg\{\gamma e^{-\beta t}\sum_{i=1}^{2} E\big[L_i(t\wedge\tau) + M_i(t\wedge\tau)\big] \\
&+ \gamma e^{-\beta t}E\big[(t\wedge\tau)\big] + e^{-\beta t}E\left[\int_0^{t\wedge\tau}\Big[\widetilde{U}(\varphi_x) + C(s)\varphi_x - U\big(C(s)\big)\Big]ds\right]\Bigg\}.
\end{aligned}$$

Maximize the right-hand side of (A.27) over $(L_i, M_i, C)_{i=1,2} \in \mathcal{A}(y_1^0, y_2^0, x^0)$ such that (A.26) holds and use (A.25) to obtain

$$\varphi(y_1^0, y_2^0, x^0) \geq \nu(t) + v(y_1^0, y_2^0, x^0) = \nu(t) + \varphi(y_1^0, y_2^0, x^0).$$

We obtain our contradiction by showing that for $t > 0$ sufficiently small, $\nu(t) > 0$.

Let us begin with the definitions

$$A_i(t) = \max_{0\leq s\leq t} B_i(s), \qquad a_i(t) = \min_{0\leq s\leq t} B_i(s).$$

According to Doob's maximal martingale inequality (e.g., [35], Theorem 1.3.8), for $\delta > 0$,

$$P\{A_i(t) \geq \delta\} \leq \frac{1}{\delta^2}E\left[A_i^2(t)\right] \leq \frac{4}{\delta^2}E\left[B_i^2(t)\right] = \frac{4t}{\delta^2},$$

and similarly, $P\{a_i(t) \leq -\delta\} \leq 4t/\delta^2$. Therefore, for all $\delta > 0$,

$$P\{A_i(t) - a_i(t) \geq \delta\} \leq P\left\{A_i(t) \geq \frac{\delta}{2}\right\} + P\left\{a_i(t) \leq -\frac{\delta}{2}\right\} \leq \frac{32t}{\delta^2}. \qquad \text{(A.28)}$$

Define the decreasing family of sets

$$F_i(t) \triangleq \left\{ |\mu_i| t + 2\sigma_i(A_i(t) - a_i(t)) \le \min\left\{ \frac{\varepsilon}{16(|y_i^0| \vee 1)}, 1 \right\} \right\}, \qquad t \ge 0.$$

Inequality (A.28) shows we can choose $T > 0$ so

$$P\{F_1(T) \cap F_2(T)\} \ge \frac{1}{2}, \qquad e^{rT} - 1 \le \frac{\varepsilon}{4\big(|x^0| \vee 1\big)}, \qquad e^{rT} \le 2. \tag{A.29}$$

From (2.1)–(2.3) we have the formulas

$$\begin{aligned} Y_i(y) &= y_i^0 + L_i(t) - M_i(t), \\ X(t) &= x^0 e^{rt} - \int_0^t e^{r(t-s)} \Big[C(s)ds + \sum_{i=1}^2 \lambda_i \left(dL_i(s) + dM_i(s) \right) \Big] \\ &\quad + \sum_{i=1}^2 \int_0^t y_i^0 e^{r(t-s)} \left(\mu_i ds + \sigma_i dB_i(s) \right) \\ &\quad + \sum_{i=1}^2 \int_0^t \int_s^t e^{r(t-\zeta)} \Big[\mu_i d\zeta + \sigma_i dB_i(\zeta) \Big] \left(dL_i(s) - dM_i(s) \right), \end{aligned}$$

from which follows

$$\begin{aligned} |Y_i(t) - y_i^0| &\le L_i(t) + M_i(t), \\ |X(t) - x^0| &\le \big(e^{rt} - 1\big)|x^0| + e^{rt} \int_0^t C(s)ds + e^{rt} \sum_{i=1}^2 \lambda_i \big[L_i(t) + M_i(t) \big] \\ &\quad + \sum_{i=1}^2 e^{rt} |y_i|^0 \Big[|\mu_i| t + \sigma_i \left(A_i(t) - a_i(t) \right) \Big] \\ &\quad + \sum_{i=1}^2 \int_0^t e^{rt} \Big[|\mu_i|(t-s) + 2\sigma_i \left(A_i(t) - a_i(t) \right) \left(dL_i(s) + dM_i(s) \right) \Big]. \end{aligned}$$

With $T > 0$ chosen to satisfy (A.29), on the set $F_1(t) \wedge F_2(t)$ we have for $0 \le t \le T$:

$$|Y_i(t) - y_i^0| \le L_i(t) + M_i(t), \tag{A.30}$$

$$|X(t) - x^0| \le \frac{\epsilon}{2} + 2\int_0^t C(s)ds + 2\sum_{i=1}^2 (1 + \lambda_i)\big(L_i(t) + M_i(t)\big). \tag{A.31}$$

From Corollary A.11 we see that $I_p(\varphi_x)$ is bounded on H, where I_p is defined by (2.16). Choose $k > \max_H I_p(\varphi_x)$ and set $\eta = \min_H \varphi_x - U'(k)$, which is strictly positive. Then (2.18) implies

$$\widetilde{U}(\varphi_x) + C(s)\varphi_x - U\big(C(s)\big) \ge \eta\big(C(s) - k\big)^+. \tag{A.32}$$

Finally, choose $t \in (0, T \wedge \frac{\varepsilon}{16k})$ and consider $\omega \in F_1(t) \wedge F_2(t)$. Define

$$\begin{aligned} Z(\omega) &= \gamma e^{-\beta t}(t \wedge \tau(\omega)) + \gamma e^{-\beta t} \sum_{i=1}^{2} \Big[L_i\big(t \wedge \tau(\omega), \omega\big) + M_i\big(t \wedge \tau(\omega), \omega\big) \Big] \\ &\quad + e^{-\beta t} \int_0^{t \wedge \tau(\omega)} \Big[\widetilde{U}\big(\varphi_x(Y_1(s,\omega), Y_2(s,\omega), X(s,\omega))\big) \\ &\qquad + C(s,\omega)\varphi_x(Y_1(s,\omega), Y_2(s,\omega), X(s,\omega)) - U(C(s,\omega)) \Big] ds. \end{aligned}$$

We consider several cases.

Case 1: $\tau(\omega) \geq t$.

We have $Z(\omega) \geq \gamma e^{-\beta t} t$.

Case 2: $\tau(\omega) < t$.

In this case, either $|X(\tau(\omega), \omega) - x_0| \geq \varepsilon$ or $|Y_1(\tau(\omega), \omega) - y_1^0| \geq \varepsilon$ or $|Y_2(\tau(\omega), \omega) - y_2^0| \geq \varepsilon$. We consider these possibilities separately.

Subcase 2A: $\tau(\omega) < t$ and $|X(\tau(\omega), \omega) - x_0| \geq \varepsilon$.

In light of (A.31), we must have either

$$2 \int_0^{\tau(\omega)} C(s,\omega) ds \geq \frac{\varepsilon}{4} \qquad \text{or} \qquad 2 \sum_{i=1}^{2} (1 + \lambda_i) \Big[L_i\big(\tau(\omega), \omega\big) + M_i\big(\tau(\omega), \omega\big) \Big] \geq \frac{\varepsilon}{4}.$$

In the latter case,

$$Z(\omega) \geq \gamma e^{-\beta t} \sum_{i=1}^{2} \Big[L_i\big(\tau(\omega), \omega\big) + M_i\big(\tau(\omega), \omega\big) \Big] \geq \gamma e^{-\beta t} \frac{\varepsilon}{8(1 + \lambda_1 \vee \lambda_2)}.$$

In the former case,

$$\begin{aligned} \frac{\varepsilon}{8} \leq \int_0^{\tau(\omega)} C(s,\omega) ds &\leq kt + \int_0^{\tau(\omega)} \big(C(s,\omega) - k\big)^+ ds \\ &\leq \frac{\varepsilon}{16} + \int_0^{\tau(\omega)} \big(C(s,\omega) - k\big)^+ ds. \end{aligned}$$

Then (A.32) implies

$$Z(\omega) \geq e^{-\beta t} \frac{\eta \varepsilon}{16}.$$

Subcase 2B: $\tau(\omega) < t$ and $\exists j \in \{1, 2\}$, $|Y_j(\tau(\omega), \omega) - y_j^0| \geq \varepsilon$.

For that j, inequality (A.30) implies

$$\Big[L_j\big(\tau(\omega), \omega\big) + M_j\big(\tau(\omega), \omega\big) \Big] \geq \varepsilon,$$

and so

$$Z(\omega) \geq \gamma e^{-\beta t}\varepsilon.$$

We have shown in every case that

$$Z(\omega) \geq e^{-\beta t} \min\left\{\gamma t, \frac{\gamma\varepsilon}{8(1+\lambda_1 \vee \lambda_2)}, \frac{\eta\varepsilon}{16}\right\} > 0 \qquad \forall \omega \in F_1(t) \wedge F_2(t),$$

and since $P\big(F_1(t) \wedge F_2(t)\big) \geq \frac{1}{2}$, we have

$$\nu(t) \geq \frac{1}{2} e^{-\beta t} \min\left\{\gamma t, \frac{\gamma\varepsilon}{8(1+\lambda_1 \vee \lambda_2)}, \frac{\eta\varepsilon}{16}\right\} > 0.$$

□

A.1 Reduction to two variables

We exploit the homogeneity of degree p of the value functions from equations (2.23)-(2.24). Direct computation of the derivatives of the value function $v(y_1, y_2, x)$ with respect to x show,

$$v_x = px^{p-1}u - x^{p-2}(u_1 y_1 + u_2 y_2,) = \left[pu - (z_1 u_1 + z_2 u_2)\right] x^{p-1}, \tag{A.33}$$

$$v_i = x^{p-1} u_i, \tag{A.34}$$

$$v_{xx} = \left[p(p-1)u + 2(1-p)(z_1 u_1 + z_2 u_2) + z_1^2 u_{11} + z_1 z_2 u_{12} + z_2^2 u_{22}\right] x^{p-2}. \tag{A.35}$$

From (2.12),(2.14) - the definitions of θ_i and A, we notice that the following holds

$$\begin{aligned}\beta - p &\left[r + \sum_{i=1}^{2} \mu_i z_i - \frac{(1-p)}{2}\left(\sigma_1^2 z_1^2 + \sigma_2^2 z_2^2 + 2\rho\sigma_1\sigma_2 z_1 z_2\right)\right] \\ &= (1-p)A + \frac{p(1-p)}{2}\left[\sigma_1^2(z_1-\theta_1)^2 + \sigma_2^2(z_2-\theta_2)^2 + 2\rho\sigma_1\sigma_2(z_1-\theta_1)(z_2-\theta_2)\right],\end{aligned} \tag{A.36}$$

we can show by additional computation that v is a classical solution to the HJB (2.20) if and only if u is a classical solution to the HJB equation (2.26).

Corollary A.15. Let $(z_1, z_2) \in \mathcal{S}_u$ and a differentiable function $\psi : \mathcal{S}_u \to R$ satisfying $\psi(z_1, z_2) = u(z_1, z_2)$ be given. If $\psi \geq u$ or $\psi \leq u$ on $\mathcal{S}_u$, then

$$p\psi(z_1, z_2) - z_1\psi_1(z_1, z_2) - z_2\psi_2(z_1, z_2) \geq pu(z_1, z_2)\Big(1 + \sum_{i=1}^{2} \lambda_i z_i\Big)^{-1} \tag{A.37}$$

PROOF: Define $\varphi(y_1, y_2, x) = x^p\psi(\frac{y_1}{x}, \frac{y_2}{x})$ for all $(y_1, y_2, x) \in \mathcal{S}_v$. If $\psi \geq u$ on $\mathcal{S}_u$ then $\varphi \geq v$ on $\mathcal{S}_v$ and similarly if $\psi \leq u$ on $\mathcal{S}_u$ then $\varphi \leq v$ on $\mathcal{S}_v$, moreover $\varphi(z_1, z_2, 1) = v(z_1, z_2, 1)$. In any case, the proof now follows from Corollary A.11 and (A.33).

□

Lemma A.16. On $\mathcal{S}_u$, u is a viscosity solution of the HJB equation (2.26).

PROOF: We prove the subsolution property, the proof for supersolution is the same. Let $(z_1, z_2) \in \mathcal{S}_u$ and $\psi \in C^2(\mathcal{S}_u)$ be given such that $\psi \geq u$ and $\psi(z_1, z_2) = u(z_1, z_2)$. Define $\varphi(y_1, y_2, x) = x^p\psi(\frac{y_1}{x}, \frac{y_2}{x})$ for all $(y_1, y_2, x) \in \mathcal{S}_v$. Then $\varphi \geq v$ on $\mathcal{S}_v$ and $\varphi(z_1, z_2, 1) = v(z_1, z_2, 1)$. Because v is a viscosity subsolution of (2.20), we have

$$\min\left\{\mathcal{L}(\varphi) - \widetilde{U}(\varphi_x); \lambda_1\varphi_x - \varphi_1; \lambda_2\varphi_x - \varphi_2; \lambda_1\varphi_x + \varphi_1; \lambda_2\varphi_x + \varphi_2\right\} \leq 0 \quad \text{at } (z_1, z_2, 1).$$

This equivalent to

$$\min\left\{\mathcal{D}(\psi) - \widetilde{U}\left(p\psi - z_1\psi_1 - z_2\psi_2\right), \mathcal{B}_1(\psi), \mathcal{S}_1(\psi), \mathcal{B}_2(\psi), \mathcal{S}_2(\psi)\right\} \leq 0 \quad \text{at } (z_1, z_2),$$

where the operators $\mathcal{D}(\psi), \mathcal{S}_i(\psi)$ and $\mathcal{B}_i(\psi)$ were defined in (2.27), (2.30) and (2.29) respectively.

The following definitions appear in [16], section 2.

Definition A.17. [16] Let $w : \mathcal{S}_u \to R$ be continuous, and $z^0 = (z_1^0, z_2^0) \in \mathcal{S}_u$. We say $(q, \mathbf{X}) \in \mathbf{J}^{2,+}_{\mathcal{S}_u} w(z^0)$ (the second-order superjet of w at z^0), if in some neighborhood of z^0,

$$w(z) \leq w(z^0) + q \cdot (z - z^0) + \frac{1}{2}(z - z^0)^T\mathbf{X}(z - z^0) + o(|z - z^0|^2) \tag{A.38}$$

By switching the inequality sign in (A.38) we arrive at the definition of the second-order subjets $\mathbf{J}^{2,-}_{\mathcal{S}_u} w(z^0)$. Moreover, define

$$\begin{aligned} \overline{\mathbf{J}}^{2,+}_{\mathcal{S}_u} w(z^0) \;=\; & \big\{(q, \mathbf{X}) \in \mathbb{R}^2 \times \mathcal{I}(2) \,|\, \exists (z_n, q_n, \mathbf{X}_n) \in \mathcal{S}_u \times \mathbb{R}^2 \times \mathcal{I}(2), \\ & \text{such that } (q_n, \mathbf{X}_n) \in \mathbf{J}^{2,+}_{\mathcal{S}_u} w(z_n) \text{ and } (z_n, q_n, \mathbf{X}_n) \to (z^0, q, \mathbf{X})\big\}, \end{aligned} \tag{A.39}$$

where $\mathcal{I}(2)$ denotes the set of all 2×2 symmetrical matrices.

Note the connection between sub- and superjets

$$\mathbf{J}^{2,-}_{\mathcal{S}_u} w(z^0) = -\mathbf{J}^{2,+}_{\mathcal{S}_u}(-w)(z^0).$$

Theorem A.18.

$$\begin{aligned} &\mathbf{J}^{2,+}_{\mathcal{S}_u} w(z^0) \\ &= \{(\nabla\varphi(z^0), \nabla^2\varphi(z^0)) | \varphi \in C^2(\mathcal{S}_u), w - \varphi \text{ has a local maximum at } z^0\} \\ &= \{(\nabla\varphi(z^0), \nabla^2\varphi(z^0)) | \varphi \in C^2(\mathcal{S}_u), \varphi(z^0) = w(z^0), \text{ and } \varphi \geq w\}. \end{aligned}$$

Proof: Fix $z^0 \in \mathcal{S}_u$, it is sufficient to prove that

$$J^{2,+}_{\mathcal{S}_u} w(z^0) \subset \{(\nabla\varphi(z^0), \nabla^2\varphi(z^0)) | \varphi \in C^2(\mathcal{S}_u), \varphi(z^0) = w(z^0), \text{ and } \varphi \geq w\}, \tag{A.40}$$

since by definition of second-order superjet, we have that

$$\begin{aligned} J^{2,+}_{\mathcal{S}_u} w(z^0) &\supset \{(\nabla\varphi(z^0), \nabla^2\varphi(z^0)) | \varphi \in C^2(\mathcal{S}_u), w - \varphi \text{ has a local maximum at } z^0\} \\ &\supset \{(\nabla\varphi(z^0), \nabla^2\varphi(z^0)) | \varphi \in C^2(\mathcal{S}_u), \varphi(z^0) = w(z^0), \text{ and } \varphi \geq w\}. \end{aligned}$$

We now prove (A.40). Let $(q, \mathbf{X}) \in J^{2,+}_{\mathcal{S}_u} w(z^0)$. For $t > 0$ define

$$\eta(t) = \sup_{0<|z-z^0|\leq t} \frac{w(z) - w(z^0) - q \cdot (z - z^0) - \frac{1}{2}(z - z^0)^T \mathbf{X}(z - z^0)}{|z - z^0|^2} \tag{A.41}$$

Then $\eta^+ = \max\{\eta, 0\}$ is a non-decreasing function and $\eta^+(t) \searrow 0$ as $t \searrow 0$, so we set $\eta(0) \triangleq 0$. We also have that

$$w(z) \leq w(z^0) + q \cdot (z - z^0) + \frac{1}{2}(z - z^0)^T \mathbf{X}(z - z^0) + \eta(|z - z^0|)|z - z^0|^2$$

Define for $t \geq 0$

$$\tilde{\eta}(t) = \int_0^t \int_0^s \eta^+(\tau) d\tau ds,$$

so that $\tilde{\eta} \in C^2([0, \infty))$ and

$$\tilde{\eta}(3t) \geq \int_{2t}^{3t} \int_t^{2t} \eta^+(\tau) d\tau ds \geq \int_{2t}^{3t} \int_t^{2t} \eta^+(t) d\tau ds \geq t^2 \eta^+(t)$$

Finally, define $\varphi(z) = w(z^0) + q \cdot (z - z^0) + \frac{1}{2}(z - z^0)^T \mathbf{X}(z - z^0) + \tilde{\eta}(3|z - z^0|)$. Then $\nabla\varphi(z^0) = q$ and $\nabla^2\varphi(z^0) = \mathbf{X}$. Finally,

$$w(z) - \varphi(z) \leq \eta(|z - z^0|)|z - z^0|^2 - \tilde{\eta}(3|z - z^0|) \leq 0 = w(z^0) - \varphi(z^0).$$

So $w(z) - \varphi(z)$ is maximized at z^0.

□

Remark A.19. Let $w : \mathcal{S}_u \to R$ be continuous, and $z^0 = (z^0_1, z^0_2) \in \mathcal{S}_u$. For$(q, \mathbf{X}) \in \mathbf{J}^{2,+}_{\mathcal{S}_u} w(z^0)$ It follows from the construction in Theorem A.18 that there exists $\varphi \in C^2(\mathcal{S}_u)$, such that $\varphi \leq w$ and $\varphi(z^0) = w(z^0)$.

Thus it follows that an equivalent definition of viscosity solution is the following

Definition A.20. [16] Let $w : \mathcal{S}_u \to R$ be continuous. We say w is a *viscosity subsolution* of (2.26) if, for every $z = (z_1, z_2) \in \mathcal{S}_u$ and for every $(q, \mathbf{X}) \in \mathbf{J}^{2,+}_{\mathcal{S}_u} w(z)$, we have $F(z, w(z), q, \mathbf{X}) \leq 0$. Similarly, w is a *viscosity supersolution* of (2.26) if, for every $z = (z_1, z_2) \in \mathcal{S}_u$ and for every $(q, \mathbf{X}) \in \mathbf{J}^{2,-}_{\mathcal{S}_u} w(z)$, we have $F(z, w(z), q, \mathbf{X}) \geq 0$.

We are now ready to prove the Comparison Theorem 4.5, which is restated below for convenience. It's proof is a modification of the proof of Theorem 3.3 from [16].

Comparison Theorem 4.5. Assume $0 < p < 1$, $\lambda_i > 0$, $i = 1, 2$ and $A > 0$. Let a continuous function $\psi^{\pm}$ be a supersolution (subsolution respectively) to (2.26) $\psi^{\pm} > 0$ in $\mathcal{S}_u$ with the boundary condition $\psi^+|_{\partial\mathcal{S}_u} \geq \psi^-|_{\partial\mathcal{S}_u} = 0$, then $\psi^+ \geq \psi^-$. In particular, any supersolution majorizes the function u defined by (2.23), and any subsolution minorizes u.

Proof: We first define some functions that permit a convenient rewrite of the HJB equation (2.26). Toward that end, we introduce the notation

$$\mathbf{C} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}, \quad \mathbf{\Gamma}(\vec{z}) = \frac{1}{2}(\mathbf{C}\vec{z}\cdot\vec{z}) \begin{bmatrix} z_1^2 & z_1 z_2 \\ z_1 z_2 & z_2^2 \end{bmatrix}, \quad \vec{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \quad \vec{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}.$$

As above, we shall denote matrices by boldface type and indicate vectors by an arrow as in $\vec{z}$. The i-th component of $\vec{z}$ is denoted z_i. The matrices $\mathbf{C}$ and $\mathbf{\Gamma}(\vec{z})$ are symmetric and positive semidefinite. We denote

$$\mathbf{\Gamma}^{\frac{1}{2}}(\vec{z}) = \begin{cases} \sqrt{\dfrac{\mathbf{C}\vec{z}\cdot\vec{z}}{2(z_1^2+z_2^2)}} \begin{bmatrix} z_1^2 & z_1 z_2 \\ z_1 z_2 & z_2^2 \end{bmatrix}, & \text{if } \vec{z} \neq 0, \\[2ex] \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & \text{if } \vec{z} = 0, \end{cases}$$

so that $(\mathbf{\Gamma}^{\frac{1}{2}}(\vec{z}))^2 = \mathbf{\Gamma}(\vec{z})$. Because the partial derivatives of $\frac{z_i^2}{\sqrt{z_1^2+z_2^2}}$ and $\frac{z_1 z_2}{\sqrt{z_1^2+z_2^2}}$ are bounded, $\mathbf{\Gamma}^{\frac{1}{2}}(\vec{z})$ is Lipschitz on $\overline{\mathcal{S}}_u$.

Define $\mathcal{O} \triangleq \{(\vec{z}, s, \vec{q}) \subset \mathcal{S}_u \times \mathbb{R}_+ \times \mathbb{R}^2 | ps - \vec{z}\cdot\vec{q} > 0\}$ and also, for $\delta > 0$ we define $\mathcal{O}_\delta \triangleq \{(\vec{z}, s, \vec{q}) \subset \mathcal{S}_u \times \mathbb{R}_+ \times \mathbb{R}^2 | ps - \vec{z}\cdot\vec{q} \geq \delta\}$. Clearly $\mathcal{O} = \bigcup_{\delta>0} \mathcal{O}_\delta$.

For $(\vec{z}, s, \vec{q}) \in \mathcal{O}$ and $\mathbf{X} \in \mathcal{I}(2)$, we define

$$\begin{aligned}
F_1(\vec{z}, s, \vec{q}) &= \lambda_1 ps - q_1 - \lambda_1 \vec{z}\cdot\vec{q}, \\
F_2(\vec{z}, s, \vec{q}) &= \lambda_2 ps - q_2 - \lambda_2 \vec{z}\cdot\vec{q}, \\
F_3(\vec{z}, s, \vec{q}) &= \lambda_1 ps + q_1 - \lambda_1 \vec{z}\cdot\vec{q}, \\
F_4(\vec{z}, s, \vec{q}) &= \lambda_2 ps + q_2 - \lambda_2 \vec{z}\cdot\vec{q}, \\
F_5(\vec{z}, s, \vec{q}, \mathbf{X}) &= F_6(\vec{z}, s) + F_7(\vec{z}, \vec{q}) + F_8(\vec{z}, \mathbf{X}) + F_9(\vec{z}, s, \vec{q}),
\end{aligned}$$

where

$$\begin{aligned}
F_6(\vec{z}, s) &= (1-p)sA + \frac{1}{2}p(1-p)s\mathbf{C}(\vec{z}-\vec{\theta})\cdot(\vec{z}-\vec{\theta}), \\
F_7(\vec{z}, \vec{q}) &= \big(r + \vec{\mu}\cdot\vec{z} - (1-p)\mathbf{C}\vec{z}\cdot\vec{z}\big)\vec{z}\cdot\vec{q}, \\
F_8(\vec{z}, \mathbf{X}) &= -\mathrm{trace}\big(\mathbf{\Gamma}(\vec{z})\mathbf{X}\big), \\
F_9(\vec{z}, s, \vec{q}) &= -\widetilde{U}(ps - \vec{z}\cdot\vec{q}).
\end{aligned}$$

Finally, we set

$$F(\vec{z}, s, \vec{q}, \mathbf{X}) = \min\big\{F_1(\vec{z}, s, \vec{q}), F_2(\vec{z}, s, \vec{q}), F_3(\vec{z}, s, \vec{q}), F_4(\vec{z}, s, \vec{q}), F_5(\vec{z}, s, \vec{q}, \mathbf{X})\big\}.$$

The HJB equation (2.26) is

$$F\big(\vec{z}, u(\vec{z}), \nabla u(\vec{z}), \nabla^{\mathbf{2}} u(\vec{z})\big) = 0.$$

We prove the comparison $\psi^+ \geq \psi^-$ on $\mathcal{S}_u$ by contradiction. Assume that $\psi^- \leq \psi^+$ does not hold everywhere in $\mathcal{S}_u$. Let $\vec{z}^0 \in \mathcal{S}_u$ be a point where $\delta_1 > 0$, the maximum of $\psi^- - \psi^+$ over $\overline{\mathcal{S}}_u$, is achieved. We use the variable doubling technique, penalizing this doubling. In particular, for each parameter $\alpha > 0$, we maximize

$$\psi^-(\vec{z}^{(1)}) - \psi^+(\vec{z}^{(2)}) - \frac{\alpha}{2}\|\vec{z}^{(1)} - \vec{z}^{(2)}\|^2$$

over $\overline{\mathcal{S}}_u \times \overline{\mathcal{S}}_u$. To this end, we define

$$M_\alpha \triangleq \sup_{(\vec{z}^{(1)}, \vec{z}^{(2)}) \in \overline{\mathcal{S}}_u \times \overline{\mathcal{S}}_u} \left\{\psi^-(\vec{z}^{(1)}) - \psi^+(\vec{z}^{(2)}) - \frac{\alpha}{2}\|\vec{z}^{(1)} - \vec{z}^{(2)}\|^2\right\}. \tag{A.42}$$

This maximum is attained at some $(\vec{z}_\alpha^{(1)}, \vec{z}_\alpha^{(2)})$ because of the continuity of $\psi^\pm$ and the compactness of $\overline{\mathcal{S}}_u$, and it satisfies

$$M_\alpha \geq \psi^-(\vec{z}^0) - \psi^+(\vec{z}^0) = \delta_1 > 0.$$

From Lemma 3.1 of [13] (see Proposition 3.7 of [13] for the proof), we have

$$\begin{cases} (i) & \lim_{\alpha\to\infty} \alpha\|\vec{z}_\alpha^{(1)} - \vec{z}_\alpha^{(2)}\|^2 = 0, \\ (ii) & \lim_{\alpha\to\infty} M_\alpha = \psi^-(\vec{z}^0) - \psi^+(\vec{z}^0) = \sup\limits_{\vec{z}\in\overline{\mathcal{S}}_u} \big(\psi^-(\vec{z}) - \psi^+(\vec{z})\big). \end{cases} \tag{A.43}$$

Using the fact that $\psi^- = 0$ on $\partial\mathcal{S}_u$, we choose a compact set $K \subset \mathcal{S}_u$ so that $\psi^- < \delta_1$ on $\overline{\mathcal{S}}_u \setminus K$. But $\psi^-(\vec{z}_\alpha^{(1)}) \geq M_\alpha \geq \delta_1$, and thus $\vec{z}_\alpha^{(1)}$ is in K for every $\alpha > 0$. Using (A.43)(i) and enlarging K if necessary, we can further guarantee that $\vec{z}_\alpha^{(2)}$ is also in K for all sufficiently large α. Finally, because $\lambda_1 z_1 + \lambda_2 z_2 + 1 = 0$ only in the complement of $\mathcal{S}_u$, for all $(z_1, z_2) \in K$ the quantity $\lambda z_1 + \lambda z_2 + 1$ is positive and hence bounded away from zero. With

$$\vec{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix},$$

we have for some $\delta_2 > 0$ that,

$$\frac{p\psi^-(\vec{z}_\alpha^{(1)})}{1 + \vec{\lambda}\cdot\vec{z}_\alpha^{(1)}} \geq \delta_2, \quad \frac{p\psi^+(\vec{z}_\alpha^{(2)})}{1 + \vec{\lambda}\cdot\vec{z}_\alpha^{(2)}} \geq \delta_2 \tag{A.44}$$

for all sufficiently large α.

We now apply Theorem 3.2 of [13] in the manner discussed in [13] immediately following the theorem. We conclude that there exist two 2×2 symmetric matrices $\mathbf{X}_\alpha$ and $\mathbf{Y}_\alpha$ such that

$$\big(\alpha(\vec{z}_\alpha^{(1)} - \vec{z}_\alpha^{(2)}), \mathbf{X}_\alpha\big) \in \overline{J}_{\mathcal{S}_u}^{2,+} \psi^-(\vec{z}_\alpha^{(1)}), \quad \big(\alpha(\vec{z}_\alpha^{(1)} - \vec{z}_\alpha^{(2)}), \mathbf{Y}_\alpha\big) \in \overline{J}_{\mathcal{S}_u}^{2,-} \psi^+(\vec{z}_\alpha^{(2)}) \tag{A.45}$$

and

$$-3\alpha \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \leq \begin{bmatrix} \mathbf{X}_\alpha & \mathbf{0} \\ \mathbf{0} & -\mathbf{Y}_\alpha \end{bmatrix} \leq 3\alpha \begin{bmatrix} \mathbf{I} & -\mathbf{I} \\ -\mathbf{I} & \mathbf{I} \end{bmatrix}, \tag{A.46}$$

where $\mathbf{I}$ is the 2×2 identity matrix. From Definition A.20, relation (A.45), and the continuity of F, it follows that

$$F\big(\vec{z}_\alpha^{(1)}, \psi^-(\vec{z}_\alpha^{(1)}), \alpha(\vec{z}_\alpha^{(1)} - \vec{z}_\alpha^{(2)}), \mathbf{X}_\alpha\big) \leq 0 \leq F\big(\vec{z}_\alpha^{(2)}, \psi^+(\vec{z}_\alpha^{(2)}), \alpha(\vec{z}_\alpha^{(1)} - \vec{z}_\alpha^{(2)}), \mathbf{Y}_\alpha\big). \tag{A.47}$$

We next argue that for α sufficiently large,

$$\big(\vec{z}_\alpha^{(1)}, \psi^-(\vec{z}_\alpha^{(1)}), \alpha(\vec{z}_\alpha^{(1)} - \vec{z}_\alpha^{(2)})\big) \in \mathcal{O}_{\delta_2}, \quad \big(\vec{z}_\alpha^{(2)}, \psi^+(\vec{z}_\alpha^{(2)}), \alpha(\vec{z}_\alpha^{(1)} - \vec{z}_\alpha^{(2)})\big) \in \mathcal{O}_{\delta_2}. \tag{A.48}$$

We fix α for which (A.44) holds and use the first membership in (A.45) to choose a sequence $(\vec{z}_{\alpha,n}, \vec{q}_{\alpha,n}, \mathbf{X}_{\alpha,n})$ converging to $(\vec{z}_\alpha^{(1)}, \alpha(\vec{z}_\alpha^{(1)} - \vec{z}_\alpha^{(2)}), \mathbf{X}_\alpha)$ and such that $(\vec{q}_{\alpha,n}, \mathbf{X}_{\alpha,n}) \in J_{\mathcal{S}_u}^{2,+} \psi^-(\vec{z}_{\alpha,n})$ for every n. Theorem A.18 implies that for each n there exists a function $\varphi_{\alpha,n} \in C^2(\mathcal{S}_u)$ such that $\varphi_{\alpha,n}(\vec{z}_{\alpha,n}) = \psi^-(\vec{z}_{\alpha,n})$, $\varphi_{\alpha,n} \geq \psi^-$, $\vec{q}_{\alpha,n} = \nabla\varphi_{\alpha,n}(\vec{z}_{\alpha,n})$, and $\mathbf{X}_{\alpha,n} = \nabla^2\varphi_{\alpha,n}(\vec{z}_{\alpha,n})$. According to Corollary A.11

$$p\psi^-(\vec{z}_{\alpha,n}) - \vec{z}_{\alpha,n} \cdot \vec{q}_{\alpha,n} = p\varphi_{\alpha,n}(\vec{z}_{\alpha,n}) - \vec{z}_{\alpha,n} \cdot \nabla\varphi_{\alpha,n}(\vec{z}_{\alpha,n}) \geq \frac{p\psi^-(\vec{z}_{\alpha,n})}{(1 + \vec{\lambda} \cdot \vec{z}_{\alpha,n})}.$$

Letting $n \to \infty$ and using (A.44), we obtain

$$p\psi^-(\vec{z}_\alpha^{(1)}) - \vec{z}_\alpha^{(1)} \cdot \alpha(\vec{z}_\alpha^{(1)} - \vec{z}_\alpha^{(2)}) \geq \frac{p\psi^-(\vec{z}_\alpha^{(1)})}{(1 + \vec{\lambda} \cdot \vec{z}_\alpha^{(1)})} \geq \delta_2.$$

By the definition of $\mathcal{O}_{\delta_2}$, the first membership in (A.48) is valid. The proof of the second membership is analogous. In conclusion,

$$p\psi^-(\vec{z}_\alpha^{(1)}) - \alpha\vec{z}_\alpha^{(1)} \cdot (\vec{z}_\alpha^{(1)} - \vec{z}_\alpha^{(2)}) \geq \delta_2, \tag{A.49}$$

$$p\psi^+(\vec{z}_\alpha^{(2)}) - \alpha\vec{z}_\alpha^{(2)} \cdot (\vec{z}_\alpha^{(1)} - \vec{z}_\alpha^{(2)}) \geq \delta_2. \tag{A.50}$$

We further conclude from (A.43)(i) and (A.50) that

$$p\psi^+(\vec{z}_\alpha^{(2)}) - \alpha\vec{z}_\alpha^{(1)} \cdot (\vec{z}_\alpha^{(1)} - \vec{z}_\alpha^{(2)}) \geq \frac{\delta_2}{2}$$

for all α sufficiently large. In other words,

$$\big(\vec{z}_\alpha^{(1)}, \psi^+(\vec{z}_\alpha^{(2)}), \alpha(\vec{z}_\alpha^{(1)} - \vec{z}_\alpha^{(2)})\big) \in \mathcal{O}_{\delta_2/2} \tag{A.51}$$

for sufficiently large α.

We verify at the end of this proof that F satisfies the following two conditions:

Condition 1: There exists $\gamma > 0$ such that

$$\gamma(s-t) \leq F(\vec{z}, s, \vec{q}, \mathbf{X}) - F(\vec{z}, t, \vec{q}, \mathbf{X}) \text{ for } s \geq t, (\vec{z}, s, \vec{q}) \in \mathcal{O}, (\vec{z}, t, \vec{q}) \in \mathcal{O}, \mathbf{X} \in \mathcal{I}(2); \quad \text{(A.52)}$$

Condition 2: For each $\delta > 0$ and bounded set $S \subset \mathbb{R}$, there exists a function $\omega\colon [0,\infty] \to [0,\infty]$ with $\omega(0+) = 0$ such that

$$\begin{aligned} & F(\vec{z}, s, \alpha(\vec{\zeta}-\vec{z}), \mathbf{Y}) - F(\vec{\zeta}, s, \alpha(\vec{\zeta}-\vec{z}), \mathbf{X}) \leq \omega\big(\alpha\|\vec{\zeta}-\vec{z}\|^2 + \|\vec{\zeta}-\vec{z}\|\big) \text{ for } s \in S, \\ & \big(\vec{z}, s, \alpha(\vec{\zeta}-\vec{z})\big) \in \mathcal{O}_\delta, \big(\vec{\zeta}, s, \alpha(\vec{\zeta}-\vec{z})\big) \in \mathcal{O}_\delta, \text{ and } \mathbf{X}, \mathbf{Y} \in \mathcal{I}(2) \text{ satisfying } (A.46). \quad \text{(A.53)} \end{aligned}$$

Using (A.47) in the fourth inequality below, and using (A.48) and (A.51) to justify the use of (A.52) in the second inequality and (A.53) in the fourth inequality, we write

$$\begin{aligned} \gamma\delta_1 &\leq \gamma\big(\psi^-(\vec{z}_\alpha^{(1)}) - \psi^+(\vec{z}_\alpha^{(2)})\big) \\ &\leq F\big(\vec{z}_\alpha^{(1)}, \psi^-(\vec{z}_\alpha^{(1)}), \alpha(\vec{z}_\alpha^{(1)} - \vec{z}_\alpha^{(2)}), \mathbf{X}_\alpha\big) - F\big(\vec{z}_\alpha^{(1)}, \psi^+(\vec{z}_\alpha^{(2)}), \alpha(\vec{z}_\alpha^{(1)} - \vec{z}_\alpha^{(2)}), \mathbf{X}_\alpha\big) \\ &\leq \big[F\big(\vec{z}_\alpha^{(1)}, \psi^-(\vec{z}_\alpha^{(1)}), \alpha(\vec{z}_\alpha^{(1)} - \vec{z}_\alpha^{(2)}), \mathbf{X}_\alpha\big) - F\big(\vec{z}_\alpha^{(2)}, \psi^+(\vec{z}_\alpha^{(2)}), \alpha(\vec{z}_\alpha^{(1)} - \vec{z}_\alpha^{(2)}), \mathbf{Y}_\alpha\big)\big] \\ &\quad + \big[F\big(\vec{z}_\alpha^{(2)}, \psi^+(\vec{z}_\alpha^{(2)}), \alpha(\vec{z}_\alpha^{(1)} - \vec{z}_\alpha^{(2)}), \mathbf{Y}_\alpha\big) - F\big(\vec{z}_\alpha^{(1)}, \psi^+(\vec{z}_\alpha^{(2)}), \alpha(\vec{z}_\alpha^{(1)} - \vec{z}_\alpha^{(2)}), \mathbf{X}_\alpha\big)\big] \\ &\leq \omega\big(\alpha\|\vec{z}_\alpha^{(1)} - \vec{z}_\alpha^{(2)}\|^2 + \|\vec{z}_\alpha^{(1)} - \vec{z}_\alpha^{(2)}\|\big). \end{aligned}$$

We now let $\alpha \to \infty$ and use (A.43)(i) to obtain a contradiction. We conclude that $\psi^- \leq \psi^+$ everywhere in $\mathcal{S}_u$.

If thus suffices to show that $F = \min_{i=1,\dots,5} F_i$ satisfies Conditions 1 and 2, and for that, it suffices to show that each F_i $i = 1, \dots, 5$, satisfies these conditions. We consider first F_1, for which Condition 1 is clearly satisfied because

$$F_1(\vec{z}, s, \vec{q}) - F_1(\vec{z}, t, \vec{q}) = \lambda_1 p(s-t).$$

Furthermore,

$$F_1\big(\vec{z}, s, \alpha(\vec{\zeta}-\vec{z})\big) - F_1\big(\vec{\zeta}, s, \alpha(\vec{\zeta}-\vec{z})\big) = \lambda_1\alpha\|\vec{\zeta}-\vec{z}\|^2$$

for all $(\vec{z}, s, \alpha(\vec{\zeta}-\vec{z}))$ and $(\vec{\zeta}, s, \alpha(\vec{\zeta}-\vec{z}))$ in $\mathcal{O}$, and thus F_1 satisfies Condition 2. Similar calculations show that F_2, F_3 and F_4 satisfy Conditions 1 and 2.

The function F_5 is the sum of the four functions F_6, F_7, F_8 and F_9. Of these four, only F_6 and F_9 are functions of s, and F_9 is increasing in s because $\widetilde{U}$ is decreasing. Therefore, for $s \geq t$,

$$\begin{aligned} F_5(\vec{z}, s, \vec{q}, \mathbf{X}) - F_5(\vec{z}, t, \vec{q}, \mathbf{X}) &= F_6(\vec{z}, s) - F_6(\vec{z}, t) + F_9(\vec{z}, s, \vec{q}) - F_9(\vec{z}, t, \vec{q}) \\ &\geq (1-p)A(s-t), \end{aligned}$$

and hence F_5 satisfies Condition 1.

We turn to the verification of Condition 2 for F_5, which we do by verifying Condition 2 for F_6, F_7, F_8 and F_9. We have the bound

$$\begin{aligned} F_6(\vec{z},s) - F_6(\vec{\zeta},s) &= \frac{1}{2}p(1-p)s\left[\mathbf{C}(\vec{z}-\vec{\theta})\cdot(\vec{z}-\vec{\theta}) - \mathbf{C}(\vec{\zeta}-\vec{\theta})\cdot(\vec{\zeta}-\vec{\theta})\right] \\ &= \frac{1}{2}p(1-p)s\left[\mathbf{C}(\vec{z}-\vec{\theta})\cdot(\vec{z}-\vec{\theta}) - \mathbf{C}(\vec{z}-\vec{\theta})\cdot(\vec{\zeta}-\vec{\theta})\right] \\ &\quad +\frac{1}{2}p(1-p)s\left[\mathbf{C}(\vec{z}-\vec{\theta})\cdot(\vec{\zeta}-\vec{\theta}) - \mathbf{C}(\vec{\zeta}-\vec{\theta})\cdot(\vec{\zeta}-\vec{\theta})\right] \\ &\le \frac{1}{2}p(1-p)s\|\mathbf{C}\|\|\vec{\zeta}-\vec{z}\|\left(\|\vec{z}-\vec{\theta}\| + \|\vec{\zeta}-\vec{\theta}\|\right) \\ &\le \text{Constant} \times \|\vec{\zeta}-\vec{z}\|, \end{aligned}$$

because $\vec{z}$ is in the compact set $\overline{\mathcal{S}}_u$ and s is in the bounded set S. The argument that F_7 satisfies

$$F_7\big(\vec{z},\alpha(\vec{\zeta}-\vec{z})\big) - F_7\big(\vec{\zeta},\alpha(\vec{\zeta}-\vec{z})\big) \le \text{Constant} \times \alpha\|\vec{\zeta}-\vec{z}\|^2$$

is only slightly more complicated and is omitted.

Turning to F_8, we have

$$F_8(\vec{z},\mathbf{Y}) - F_8(\vec{\zeta},\mathbf{X}) = \text{trace}\big(\mathbf{\Gamma}(\vec{\zeta})\mathbf{X} - \mathbf{\Gamma}(\vec{z})\mathbf{Y}\big) = \text{trace}\big(\mathbf{\Gamma}^{\frac{1}{2}}(\vec{\zeta})\mathbf{X}\mathbf{\Gamma}^{\frac{1}{2}}(\vec{\zeta}) - \mathbf{\Gamma}^{\frac{1}{2}}(\vec{z})\mathbf{Y}\mathbf{\Gamma}^{\frac{1}{2}}(\vec{z})\big) \tag{A.54}$$

because trace($\mathbf{AB}$) = trace($\mathbf{BA}$) for square matrices $\mathbf{A}$ and $\mathbf{B}$. But the second inequality in (A.46) means that

$$\begin{bmatrix} 3\alpha\mathbf{I} - \mathbf{X}_\alpha & -3\alpha\mathbf{I} \\ -3\alpha\mathbf{I} & 3\alpha\mathbf{I} + \mathbf{Y}_\alpha \end{bmatrix}$$

is positive semidefinite, which implies that for every pair of two-dimensional vectors $\vec{v}$ and $\vec{w}$,

$$0 \le [\vec{v}^{tr}\vec{w}^{tr}]\begin{bmatrix} 3\alpha\mathbf{I} - \mathbf{X} & -3\alpha\mathbf{I} \\ -3\alpha\mathbf{I} & 3\alpha\mathbf{I} + \mathbf{Y} \end{bmatrix}\begin{bmatrix} \vec{v} \\ \vec{w} \end{bmatrix} = 3\alpha\|\vec{v}-\vec{w}\|^2 - (\vec{v}^{tr}\mathbf{X}\vec{v} - \vec{w}^{tr}\mathbf{Y}\vec{w}). \tag{A.55}$$

We first apply inequality (A.55) with $\vec{v}$ equal to the first column (which is also the first row) of $\mathbf{\Gamma}^{\frac{1}{2}}(\vec{\zeta})$ and $\vec{w}$ equal to the first column (which is also the first row) of $\mathbf{\Gamma}^{\frac{1}{2}}(\vec{z})$ to conclude that the $(1,1)$ entry of $\mathbf{\Gamma}^{\frac{1}{2}}(\vec{\zeta})\mathbf{X}\mathbf{\Gamma}^{\frac{1}{2}}(\vec{\zeta}) - \mathbf{\Gamma}^{\frac{1}{2}}(\vec{z})\mathbf{Y}\mathbf{\Gamma}^{\frac{1}{2}}(\vec{z})$ is dominated by $3\alpha L\|\vec{\zeta}-\vec{z}\|^2$, where L is the constant associated with the Lipschitz continuity of $\mathbf{\Gamma}^{\frac{1}{2}}(\cdot)$. We next apply (A.55) with $\vec{v}$ equal to the second column of $\mathbf{\Gamma}^{\frac{1}{2}}(\vec{\zeta})$ and $\vec{w}$ equal to the second column of $\mathbf{\Gamma}^{\frac{1}{2}}(\vec{z})$ to conclude that the $(2,2)$ entry of $\mathbf{\Gamma}^{\frac{1}{2}}(\vec{\zeta})\mathbf{X}\mathbf{\Gamma}^{\frac{1}{2}}(\vec{\zeta}) - \mathbf{\Gamma}^{\frac{1}{2}}(\vec{z})\mathbf{Y}\mathbf{\Gamma}^{\frac{1}{2}}(\vec{z})$ is dominated by $3\alpha L\|\vec{\zeta}-\vec{z}\|^2$. Summing these two equalities, we see that

$$F_8(\vec{z},\mathbf{Y}) - F_8(\vec{\zeta},\mathbf{X}) = \text{trace}\big(\mathbf{\Gamma}^{\frac{1}{2}}(\vec{\zeta})\mathbf{X}\mathbf{\Gamma}^{\frac{1}{2}}(\vec{\zeta}) - \mathbf{\Gamma}^{\frac{1}{2}}(\vec{z})\mathbf{Y}\mathbf{\Gamma}^{\frac{1}{2}}(\vec{z})\big) \le 6\alpha L\|\vec{\zeta}-\vec{z}\|^2$$

whenever $\mathbf{X}$ and $\mathbf{Y}$ satisfy (A.46). In other words, F_8 satisfies Condition 2.

Finally, assume $(\vec{z}, s, \alpha(\vec{\zeta} - \vec{z}))$ and $(\vec{\zeta}, s, \alpha(\vec{\zeta} - \vec{z}))$ are in $\mathcal{O}_\delta$. The Mean-Value Theorem implies

$$\begin{aligned} F_9\big(\vec{z}, s, \alpha(\vec{\zeta} - \vec{z})\big) - F_9\big(\vec{\zeta}, s, \alpha(\vec{\zeta} - \vec{z})\big) &= \widetilde{U}\big(ps - \alpha\vec{\zeta} \cdot (\vec{\zeta} - \vec{z})\big) - \widetilde{U}\big(ps - \alpha\vec{z} \cdot (\vec{\zeta} - \vec{z})\big) \\ &= -\widetilde{U}'(\eta)\alpha\|\vec{\zeta} - \vec{z}\|^2, \end{aligned}$$

where η is between $ps - \alpha\vec{\zeta} \cdot (\vec{\zeta} - \vec{z})$ and $ps - \alpha\vec{z} \cdot (\vec{\zeta} - \vec{z})$. Both these quantities are greater than or equal to δ and hence $\eta \geq \delta$. Since $\widetilde{U}'$ is increasing and negative, $-\widetilde{U}'(\eta) \leq -\widetilde{U}'(\delta)$, and consequently F_9 satisfies Condition 2.

□

B Appendix - Regularity of $u(z_1, z_2)$

In this section we assume that the solvency region is divided into nine regions. We will abuse notation and call them $D, N, S, E, W, SW, SE, NE$ and NW regions. They are not necessarily the optimal regions. Our goal is to extend $w^D \in C^2(D)$ continuously from $\overline{D}$ into the rest of the solvency region using the four first-order equations. Specifically, we wish to define $w \in C(\mathcal{S}_u)$ so that $w|_D = w^D$ and such that in the southern region $\mathcal{B}_2(w) = 0$, and in the southwestern region $\mathcal{B}_2(w) = \mathcal{B}_1(w) = 0$, and so on as presented in Table 1 in Section 3. This result is stated in Theorem B.2. The intended application of this theory is to extend the constructed sub- and supersolution, defined in $\overline{D}$, to the rest of the solvency region. In this section we concentrate on the southern and the southwestern regions. The treatment of all other regions is analogous.

As a reminder, the four first-order equations are:

$$\mathcal{B}_1(w) = \lambda_1 pw + (-1 - \lambda_1 z_1)w_1 - \lambda_1 z_2 w_2 = 0, \tag{B.1}$$
$$\mathcal{B}_2(w) = \lambda_2 pw + (-1 - \lambda_2 z_2)w_2 - \lambda_2 z_1 w_1 = 0, \tag{B.2}$$
$$\mathcal{S}_1(w) = \lambda_1 pw + (1 - \lambda_1 z_1)w_1 - \lambda_1 z_2 w_2 = 0, \tag{B.3}$$
$$\mathcal{S}_2(w) = \lambda_2 pw + (1 - \lambda_2 z_2)w_2 - \lambda_2 z_1 w_1 = 0. \tag{B.4}$$

Assumption B.1.

(i) The D region has four corners and there are smooth parameterizations of the four sides of the boundary of D. For example, let us denote the southwestern corner and the southeastern corner of D by $(z_{1,SW}, z_{2,SW})$ and $(z_{1,SE}, z_{2,SE})$, respectively. Then there is a C^2 function f such that the boundary between $(z_{1,SW}, z_{2,SW})$ and $(z_{1,SE}, z_{2,SE})$ is parameterized by

$$\widehat{z}_2 = f(\widehat{z}_1), \quad z_{2,SW} \leq \widehat{z}_1 \leq z_{2,SE}, \tag{B.5}$$

such that

$$z_{2,SW} = f(z_{1,SW}), \quad z_{2,SE} = f(z_{1,SE}).$$

(ii) $w^D \in C^2(\overline{D})$, where the derivatives on the boundary are defined as limits from inside D of the appropriate of its first- and second-order partial derivatives. In particular, these limits are defined on the boundaries of D. Furthermore, these limits are the partial derivatives at the boundaries computed along directions into D.

Theorem B.2. Under the Assumptions B.1, there exists a unique extension w of w^D from $\overline{D}$ to the $\overline{S}$ and the $\overline{SW}$ regions satisfying $\mathcal{B}_2(w) = 0$ in the S region and $\mathcal{B}_2(w^S) = 0 = \mathcal{B}_1(w^D)$ in SW and the other conditions from Table 1. We also have that w is zero on $\partial \mathcal{S}_u$. Moreover, if $\mathcal{B}_2(w^D) = 0$ on $\partial D \cap \partial S$, $\mathcal{S}_2(w^D) = 0$ on $\partial D \cap \partial N$, $\mathcal{S}_1(w^D) = 0$ on $\partial D \cap \partial W$ and $\mathcal{B}_1(w^D) = 0$ on $\partial D \cap \partial E$, then $w \in C^1(\mathcal{S}_u)$.

In order to prove this theorem, we first define the extension of w^D in the southern and southwestern regions. We then calculate its derivatives and prove that under the appropriate conditions these derivatives would be continuous across boundaries of these regions.

B.1 South-Western Region

The reader can verify that given $w^D(z_{1,SW}, z_{2,SW})$, for $(z_1, z_2) \in \overline{SW}$ if we define w^{SW} according to equation (3.14); i.e.,

$$w^{SW}(z_1, z_2) = \left(\frac{1 + \lambda_1 z_1 + \lambda_2 z_2}{1 + \lambda_1 z_{1,SW} + \lambda_2 z_{2,SW}} \right)^p w^D(z_{1,SW}, z_{2,SW}), \tag{B.6}$$

then $\mathcal{B}_2(w^{SW}) = \mathcal{B}_1(w^{SW}) = 0$ as desired.

Remark B.3. Note that

$$\lim_{\substack{(z_1, z_2) \in SW \\ (z_1, z_2) \to (z_{1,SW}, z_{2,SW})}} w^{SW}(z_1, z_2) = w^D(z_{1,SW}, z_{2,SW}).$$

B.2 Southern Region

In S we want the equation $\mathcal{B}_2(w) = 0$ to hold. We solve this equation using characteristics. Assume that for $s = 0$ we start at the southern boundary of D and denote that point

$$(\widehat{z}_1, \widehat{z}_2) = (z_1(0), z_2(0)). \tag{B.7}$$

Let $\dot{z}_2(s) = -(1+\lambda_2 z_2(s))$, $\dot{z}_1(s) = -\lambda_2 z_1(s)$, so that $\mathcal{B}_2(w^D) = 0$ implies $\frac{d}{ds}w(z(s)) = -\lambda_2 p w(z(s))$. Therefore

$$\begin{aligned} z_2(s) &= \frac{-1}{\lambda_2} + \left(\widehat{z}_2 + \frac{1}{\lambda_2}\right) e^{-\lambda_2 s}, && \text{(B.8)} \\ z_1(s) &= \widehat{z}_1 e^{-\lambda_2 s}, && \text{(B.9)} \\ w(z(s)) &= w(\widehat{z}_1, \widehat{z}_2) e^{-\lambda_2 p s}. && \text{(B.10)} \end{aligned}$$

Suppressing the dependency on s in the notation of (z_1, z_2), in (B.8) and (B.9), we have

$$e^{-\lambda_2 s} = \frac{z_1}{\widehat{z}_1} = \frac{1+\lambda_2 z_2}{1+\lambda_2 \widehat{z}_2},$$

and we define $w^S(z_1, z_2) = w(z(s))$ as the unique solution to (B.10) with the initial condition $w(z(0)) = w^D(\widehat{z}_1, \widehat{z}_2)$, which is

$$w^S(z_1, z_2) = \left(\frac{1+\lambda_2 z_2}{1+\lambda_2 \widehat{z}_2}\right)^p w^D(\widehat{z}_1, \widehat{z}_2). \tag{B.11}$$

The characteristic line's equation is $z_1 = \frac{\lambda_2 \widehat{z}_1}{1+\lambda_2 \widehat{z}_2} z_2 + \frac{\widehat{z}_1}{1+\lambda_2 \widehat{z}_2}$, or in a more convenient form $\frac{1+\lambda_2 z_2}{1+\lambda_2 \widehat{z}_2} = \frac{z_1}{\widehat{z}_1}$. Notice that all the characteristics intersect at $(0, -\frac{1}{\lambda_2})$. We now, go back to Assumption B.1.(i) and we see that in the definition $\widehat{z}_2 = f(\widehat{z}_1)$ in (B.5) the real meaning of $(\widehat{z}_1, \widehat{z}_2)$ is $(z_1(0), z_2(0))$, i.e. the boundary points which are the origins of the characteristic lines. Based on the discussion above, we see that $w^S(z_1, z_2)$ defined by (B.11) solves $\mathcal{B}_2(w^S) = 0$. Note, that we have also proved the following slightly more general proposition:

Proposition B.4. Fix $(z_1(0), z_2(0))$. The characteristic curve $(z_1(s), z_2(s))$, $0 \le s < \infty$, of the equation $\mathcal{B}_2(\psi) = 0$ that passes through $(z_1(0), z_2(0))$ is the straight line passing also through the southern corner of the solvency region $(0, -\frac{1}{\lambda_2})$ and is given by $\frac{1+\lambda_2 z_2(t)}{1+\lambda_2 z_2(0)} = \frac{z_1(t)}{z_1(0)}$. Specifically, if $(\widehat{z}_1, \widehat{z}_2) = (z_1(0), z_2(0))$ and we denote $(z_1, z_2) = (z_1(t), z_2(t))$, then the characteristic equation becomes

$$\frac{1+\lambda_2 z_2}{1+\lambda_2 \widehat{z}_2} = \frac{z_1}{\widehat{z}_1}. \tag{B.12}$$

From (B.12) it also follows that

$$\begin{aligned} \frac{1+\lambda_2 \widehat{z}_2 - \lambda_2 \widehat{z}_1 f'(\widehat{z}_1)}{1+\lambda_2 z_2 - \lambda_2 z_1 f'(\widehat{z}_1)} &= \frac{1+\lambda_2 \widehat{z}_2 - \lambda_2 \widehat{z}_1 f'(\widehat{z}_1)}{1+\lambda_2 \widehat{z}_2 \frac{z_1}{\widehat{z}_1} - \lambda_2 z_1 f'(\widehat{z}_1)} \\ &= \frac{\widehat{z}_1 \left(1+\lambda_2 \widehat{z}_2 - \lambda_2 \widehat{z}_1 f'(\widehat{z}_1)\right)}{(1+\lambda_2 \widehat{z}_1) z_1 - \lambda_2 z_1 f'(\widehat{z}_1)} = \frac{z_1}{\widehat{z}_1}. \end{aligned} \tag{B.13}$$

Furthermore, if ϕ satisfies (B.2) along this line, then for $s \ge t$, we have $\phi(z_1(s), z_2(s)) = \left(\frac{1+\lambda_2 z_2(s)}{1+\lambda_2 z_2(t)}\right)^p \phi(z_1(t), z_2(t))$. Similar result holds for the northern, eastern and western regions.

We need to add one more assumption

Assumption B.1

(iii) Assume that the characteristic line starting at $(\widehat{z}_1, \widehat{z}_2)$ does not cross the southern boundary at any other point.

Remark B.5. If $w^D(\widehat{z}_1, \widehat{z}_2) = \frac{A^{p-1}}{p} + O(\lambda^{\frac{2}{3}})$, then from (B.11) and (B.12) we see that $\frac{w^S(z_1,z_2)}{z_1^p} = \frac{1}{\widehat{z}_1^p} w^D(\widehat{z}_1, \widehat{z}_2) = \frac{A^{p-1}}{p\widehat{z}_1^p} + O(\lambda^{\frac{2}{3}})$.

Remark B.6. When $(z_1, z_2) \in \partial S \cap \partial SW$ it follows that $(\widehat{z}_1, \widehat{z}_2) = (z_{1,SW}, z_{2,SW})$, and

$$\frac{1 + \lambda_2 z_2}{1 + \lambda_2 z_{2,SW}} = \frac{z_1}{z_{1,SW}} = \frac{1 + \lambda_1 z_1 + \lambda_2 z_2}{1 + \lambda_1 z_{1,SW} + \lambda_2 z_{2,SW}}. \tag{B.14}$$

This is because the two equalities in equation (B.14) are that of an equation of a line passing through points $(z_{1,SW}, z_{2,SW})$ and $(0, -\frac{1}{\lambda_2})$. Thus from (B.6) and (B.11), it follows that for $(z_1, z_2) \in \partial S \cap \partial SW$,

$$\begin{aligned} w^S(z_1, z_2) &= \left(\frac{1 + \lambda_2 z_2}{1 + \lambda_2 z_{2,SW}}\right)^p w^D(z_{1,SW}, z_{2,SW}) \\ &= \left(\frac{1 + \lambda_1 z_1 + \lambda_2 z_2}{1 + \lambda_1 z_{1,SW} + \lambda_2 z_{2,SW}}\right)^p w^D(z_{1,SW}, z_{2,SW}) = w^{SW}(z_1, z_2) \end{aligned}$$

□

Remark B.7. Let $(\overline{z_1}, \overline{z_2}) \in \partial S \cap \partial D$ be a point on the southern boundary. Then because of the chosen initial condition in (B.11), it follows that

$$\lim_{\substack{(z_1, z_2) \in S \\ (z_1, z_2) \to (\overline{z_1}, \overline{z_2})}} w^S(z_1, z_2) = w^D(\overline{z_1}, \overline{z_2}). \tag{B.15}$$

□

Remark B.8. Under Assumptions B.1, note that for $(z_1, z_2) \in S$ we have that $w^S(z_1, z_2)$ is continuously differentiable in the direction of the characteristic line $\begin{pmatrix} \lambda_2 z_1 \\ 1 + \lambda_2 z_2 \end{pmatrix}$.

Remark B.9. Fix $(z_1, z_2) \in SW$. It follows from Proposition B.4 that the line connecting $(z_{1,SW}, z_{2,SW})$ and $(-\frac{1}{\lambda_1}, 0)$ is a characteristic line of equation (B.1), and similarly the line connecting (z_1, z_2) with $(0, -\frac{1}{\lambda_2})$ is a characteristic line of equation (B.2). Define (z_1^0, z_2^0) as the point of their intersection. Since (z_1^0, z_2^0) is on the line passing through $(z_{1,SW}, z_{2,SW})$ and $(-\frac{1}{\lambda_1}, 0)$, we have $\frac{1+\lambda_1 z_1^0}{1+\lambda_1 z_{1,SW}} = \frac{z_2^0}{z_{2,SW}}$. It follows that

$$1 + \lambda_1 z_{1,SW} + \lambda_2 z_{2,SW} = 1 + \lambda_1 z_{1,SW} + \lambda_2 \left(\frac{1 + \lambda_1 z_{1,SW}}{1 + \lambda_1 z_1^0}\right) z_2^0 = \frac{1 + \lambda_1 z_{1,SW}}{1 + \lambda_1 z_1^0}(1 + \lambda_1 z_1^0 + \lambda_2 z_2^0). \tag{B.16}$$

Similarly, because (z_1, z_2) is on the line passing through (z_1^0, z_2^0) and $(0, -\frac{1}{\lambda_1})$, we have $\frac{1+\lambda_2 z_2}{1+\lambda_2 z_2^0} = \frac{z_1}{z_1^0}$. It follows that

$$1 + \lambda_1 z_1 + \lambda_2 z_2 = 1 + \lambda_1 \left(\frac{1 + \lambda_2 z_2}{1 + \lambda_2 z_2^0}\right) z_1^0 + \lambda_2 z_2 = \frac{1 + \lambda_2 z_2}{1 + \lambda_2 z_2^0}(1 + \lambda_1 z_1^0 + \lambda_2 z_2^0). \tag{B.17}$$

Combining (B.16) with (B.17) we get

$$\frac{1+\lambda_2 z_2}{1+\lambda_2 z_2^0}\frac{1+\lambda_1 z_1^0}{1+\lambda_1 z_{1,SW}} = \frac{1+\lambda_1 z_1+\lambda_2 z_2}{1+\lambda_1 z_{1,SW}+\lambda_2 z_{2,SW}}. \tag{B.18}$$

Proposition B.4, applied once for characteristics of (B.2), and once more for characteristics of (B.1), gives us

$$\begin{aligned} w^{SW}(z_1,z_2) &= \left(\frac{1+\lambda_2 z_2}{1+\lambda_2 z_2^0}\right)^p w^{SW}(z_1^0,z_2^0) \qquad \text{(B.19)} \\ &= \left(\frac{1+\lambda_2 z_2}{1+\lambda_2 z_2^0}\right)^p \left(\frac{1+\lambda_1 z_1^0}{1+\lambda_1 z_{1,SW}}\right)^p w^D(z_{1,SW},z_{2,SW}), \end{aligned}$$

which is consistent with (B.6) because of (B.18). □

Remark B.10. It follows from Remarks B.3 and B.7 that the extension w^D agrees with w^S and with w^{SW} on the southern boundary of the D region $\partial S \cap \partial D$ and at $(z_{1,SW}, z_{2,SW})$ respectively, and w^S agrees with w^{SW} on $\partial S \cap \partial SW$. The extensions w^S and w^{SW} are continuous up to and including the boundaries. In a similar way, we extend w^D from D to the entire solvency region. Combining this together we have extended w^D from D to the rest of solvency region as a continuous function. We refer to this extended function as w.

□

Proposition B.11. If w^D satisfies Assumptions B.1, then the extended function w, extended by the method described above, is continuous in the entire solvency region $\mathcal{S}_u$. Moreover $w|_{\partial\mathcal{S}_u} = 0$.

Proof: Remark B.10 establishes the continuity of w. To see that $w|_{\partial\mathcal{S}_u} = 0$ consider

Case 1: $(z_1^0, z_2^0) \in \partial S \cap \partial\mathcal{S}_u$.

Note that $(z_1^0, z_2^0) = (0, -\frac{1}{\lambda_2})$ is the only point on intersection of the boundaries of the southern and solvency regions. Therefore

$$w^S(z_1^0,z_2^0) = \left(\frac{1+\lambda_2 z_2^0}{1+\lambda_2 \widehat{z}_2}\right)^p w^D(\widehat{z}_1,\widehat{z}_2) = 0.$$

Case 2: $(z_1^0, z_2^0) \in \partial SW \cap \partial\mathcal{S}_u$.

In this case (z_1^0, z_2^0) is a point on the line connecting the two corners of the solvency region $(0, -\frac{1}{\lambda_2})$ and $(-\frac{1}{\lambda_1}, 0)$. Then

$$w^{SW}(z_1^0,z_2^0) = \left(\frac{1+\lambda_1 z_1^0+\lambda_2 z_2^0}{1+\lambda_1 z_{1,SW}+\lambda_2 z_{2,SW}}\right)^p w^D(z_{1,SW},z_{2,SW}) = 0.$$

The proof for the rest of $\partial\mathcal{S}_u$ is the same.

□

B.3 First derivatives

We show that for $(z_1, z_2) \in S$ we have that

$$\frac{\partial \widehat{z}_1}{\partial z_2} = -\frac{\lambda_2 \widehat{z}_1}{1 + \lambda_2 z_2 - \lambda_2 z_1 f'(\widehat{z}_1)}, \tag{B.20}$$

$$\frac{\partial \widehat{z}_1}{\partial z_1} = \frac{1 + \lambda_2 \widehat{z}_2}{1 + \lambda_2 z_2 - \lambda_2 z_1 f'(\widehat{z}_1)} = -\frac{1 + \lambda_2 \widehat{z}_2}{\lambda_2 \widehat{z}_1} \frac{\partial \widehat{z}_1}{\partial z_2}, \tag{B.21}$$

$$\frac{\partial \widehat{z}_2}{\partial z_2} = f'(\widehat{z}_1) \frac{\partial \widehat{z}_1}{\partial z_2}, \tag{B.22}$$

$$\frac{\partial \widehat{z}_2}{\partial z_1} = f'(\widehat{z}_1) \frac{1 + \lambda_2 \widehat{z}_2}{1 + \lambda_2 z_2 - \lambda_2 z_1 f'(\widehat{z}_1)} = -f'(\widehat{z}_1) \frac{1 + \lambda_2 \widehat{z}_2}{\lambda_2 \widehat{z}_1} \frac{\partial \widehat{z}_1}{\partial z_2}. \tag{B.23}$$

Indeed, to derive (B.20) we use (B.12) to write

$$\widehat{z}_1 = \frac{(1 + \lambda_2 \widehat{z}_2) z_1}{1 + \lambda_2 z_2} = \frac{(\lambda_2 f(\widehat{z}_1) + 1) z_1}{1 + \lambda_2 z_2}, \tag{B.24}$$

so

$$\begin{aligned} \frac{\partial \widehat{z}_1}{\partial z_2} &= -\frac{(\lambda_2 f(\widehat{z}_1) + 1) \lambda_2 z_1}{(1 + \lambda_2 z_2)^2} + \frac{\lambda_2 f'(\widehat{z}_1) z_1}{1 + \lambda_2 z_2} \frac{\partial \widehat{z}_1}{\partial z_2} \\ &= -\frac{\lambda_2 \widehat{z}_1}{1 + \lambda_2 z_2} + \frac{\lambda_2 z_1 f'(\widehat{z}_1)}{1 + \lambda_2 z_2} \frac{\partial \widehat{z}_1}{\partial z_2}. \end{aligned}$$

Therefore

$$\frac{1 + \lambda_2 z_2 - \lambda_2 z_1 f'(\widehat{z}_1)}{1 + \lambda_2 z_2} \frac{\partial \widehat{z}_1}{\partial z_2} = -\frac{\lambda_2 \widehat{z}_1}{1 + \lambda_2 z_2},$$

which implies (B.20).

For (B.21), we obtain from (B.24)

$$\frac{\partial \widehat{z}_1}{\partial z_1} = \frac{\lambda_2 f(\widehat{z}_1) + 1}{1 + \lambda_2 z_2} + \frac{\lambda_2 f'(\widehat{z}_1) z_1}{1 + \lambda_2 z_2} \frac{\partial \widehat{z}_1}{\partial z_1},$$

which is equivalent to

$$\frac{1 + \lambda_2 z_2 - \lambda_2 z_1 f'(\widehat{z}_1)}{1 + \lambda_2 z_2} \frac{\partial \widehat{z}_1}{\partial z_1} = \frac{1 + \lambda_2 \widehat{z}_2}{1 + \lambda_2 z_2},$$

and (B.21) follows.

To derive (B.22) observe that

$$\frac{\partial \widehat{z}_2}{\partial z_2} = \frac{\partial}{\partial z_2} f(\widehat{z}_1) = f'(\widehat{z}_1) \frac{\partial \widehat{z}_1}{\partial z_2},$$

and (B.23) follows similarly.

Remark B.12. Note that both $\frac{\partial \widehat{z}_1}{\partial z_2}$ and $\frac{\partial \widehat{z}_2}{\partial z_2}$ are of order $O(\lambda)$. Also $\frac{\partial \widehat{z}_1}{\partial z_1} = 1 + O(\lambda)$ and $\frac{\partial \widehat{z}_2}{\partial z_1} = f'(\widehat{z}_1) + O(\lambda)$. □

Remark B.13. Note that from (B.12), we have $\frac{z_1}{1+\lambda_2 z_2} = \frac{\widehat{z}_1}{1+\lambda_2 \widehat{z}_2}$. Using this the reader can verify that $z_1 \frac{\partial \widehat{z}_i}{\partial z_j}$, $i, j = 1, 2$ depends on (z_1, z_2) only through $(\widehat{z}_1, \widehat{z}_2)$. □

Differentiating (B.11), and get

$$\begin{aligned} w_2^S(z_1, z_2) &= \frac{\lambda_2 p}{1+\lambda_2 z_2}\left(\frac{1+\lambda_2 z_2}{1+\lambda_2 \widehat{z}_2}\right)^p w(\widehat{z}_1, \widehat{z}_2) - \frac{\lambda_2 p (1+\lambda_2 z_2)^p}{(1+\lambda_2 \widehat{z}_2)^{p+1}} \frac{\partial \widehat{z}_2}{\partial z_2} w(\widehat{z}_1, \widehat{z}_2) \\ &\quad + \left(\frac{1+\lambda_2 z_2}{1+\lambda_2 \widehat{z}_2}\right)^p \left(w_2^D(\widehat{z}_1, \widehat{z}_2) \frac{\partial \widehat{z}_2}{\partial z_2} + w_1^D(\widehat{z}_1, \widehat{z}_2) \frac{\partial \widehat{z}_1}{\partial z_2} \right) \\ &= \frac{\lambda_2 p}{1+\lambda_2 z_2} \left(\frac{z_1}{\widehat{z}_1}\right)^p w(\widehat{z}_1, \widehat{z}_2) \\ &\quad - \left(\frac{z_1}{\widehat{z}_1}\right)^p \frac{\partial \widehat{z}_1}{\partial z_2} \left[\frac{\lambda_2 p f'(\widehat{z}_1)}{1+\lambda_2 \widehat{z}_2} w(\widehat{z}_1, \widehat{z}_2) - f'(\widehat{z}_1) w_2^D(\widehat{z}_1, \widehat{z}_2) - w_1^D(\widehat{z}_1, \widehat{z}_2) \right], \end{aligned} \tag{B.25}$$

$$\begin{aligned} w_1^S(z_1, z_2) &= -\frac{\lambda_2 p (1+\lambda_2 z_2)^p}{(1+\lambda_2 \widehat{z}_2)^{p+1}} \frac{\partial \widehat{z}_2}{\partial z_1} w(\widehat{z}_1, \widehat{z}_2) \\ &\quad + \left(\frac{1+\lambda_2 z_2}{1+\lambda_2 \widehat{z}_2}\right)^p \left(w_2^D(\widehat{z}_1, \widehat{z}_2) \frac{\partial \widehat{z}_2}{\partial z_1} + w_1^D(\widehat{z}_1, \widehat{z}_2) \frac{\partial \widehat{z}_1}{\partial z_1} \right) \\ &= \left(\frac{z_1}{\widehat{z}_1}\right)^p \frac{\partial \widehat{z}_1}{\partial z_2} \left[\frac{\lambda_2 p f'(\widehat{z}_1)}{\lambda_2 \widehat{z}_1} w(\widehat{z}_1, \widehat{z}_2) - \frac{1+\lambda_2 \widehat{z}_2}{\lambda_2 \widehat{z}_1} \left(f'(\widehat{z}_1) w_2^D(\widehat{z}_1, \widehat{z}_2) + w_1^D(\widehat{z}_1, \widehat{z}_2) \right) \right]. \end{aligned} \tag{B.26}$$

It is straight forward to verify from (B.11), (B.12), (B.25) and (B.26) that w^S satisfies

$$\mathcal{B}_2(w^S) = 0 \tag{B.27}$$

in S, as it was constructed to do.

Lemma B.14. Assume that $w^D(\widehat{z}_1, \widehat{z}_2) = \frac{A^{p-1}}{p} + O(\lambda^{\frac{2}{3}})$ and that $w_i^D(\widehat{z}_1, \widehat{z}_2) = O(\lambda)$. Then $\frac{1}{\lambda_2 z_1^{p-1}} w_2^S(z_1, z_2)$ depends on (z_1, z_2) only through $(\widehat{z}_1, \widehat{z}_2)$, and

$$\frac{1}{\lambda_2 z_1^{p-1}} w_2^S(z_1, z_2) = \frac{A^{p-1}}{\widehat{z}_1^{p-1}} + O\left(\lambda^{\frac{2}{3}}\right). \tag{B.28}$$

Proof: Indeed, under the assumption $w_i^D = O(\lambda)$ it follows from equation (B.25) that

$$\begin{aligned} \frac{1}{\lambda_2} \frac{1}{z_1^{p-1}} w_2^S(z_1, z_2) &= \frac{p z_1}{1+\lambda_2 z_2} \left(\frac{1}{\widehat{z}_1}\right)^p w(\widehat{z}_1, \widehat{z}_2) \\ &+ \left(\frac{1}{\widehat{z}_1}\right)^p \frac{\widehat{z}_1 z_1}{1+\lambda_2 z_2 - \lambda_2 z_1 f'(\widehat{z}_1)} \left[\frac{\lambda_2 p f'(\widehat{z}_1)}{1+\lambda_2 \widehat{z}_2} w(\widehat{z}_1, \widehat{z}_2) - f'(\widehat{z}_1) w_2^D(\widehat{z}_1, \widehat{z}_2) - w_1^D(\widehat{z}_1, \widehat{z}_2) \right] \end{aligned} \tag{B.29}$$

Notice that the right hand side only depends on (z_1, z_2) only through $(\widehat{z}_1, \widehat{z}_2)$, since from (B.12) $\frac{z_1}{1+\lambda_2 z_2} = \frac{\widehat{z}_1}{1+\lambda_2 \widehat{z}_2}$. It now follows that

$$\frac{1}{\lambda_2 z_1^{p-1}} w_2^S(z_1, z_2) = \frac{p}{\widehat{z}_1^{p-1}} w(\widehat{z}_1, \widehat{z}_2) + O(\lambda) = \frac{A^{p-1}}{\widehat{z}_1^{p-1}} + O\left(\lambda^{\frac{2}{3}}\right),$$

with $O\left(\lambda^{\frac{2}{3}}\right)$ independent of (z_1, z_2) on the same characteristic line.

□

Lemma B.15. Under the Assumptions B.1, fix $(\overline{z}_1, \overline{z}_2) \in \partial D \cap \partial S$. If $\mathcal{B}_2(w^D)(\overline{z}_1, \overline{z}_2) = 0$, then

$$\lim_{\substack{(z_1, z_2) \in \overline{S} \\ (z_1, z_2) \to (\overline{z}_1, \overline{z}_2)}} w_1^S(z_1, z_2) = w_1^D(\overline{z}_1, \overline{z}_2).$$

PROOF: From (B.26), Proposition B.11, and using $w^D \in C^1(\overline{D})$, we have

$$\begin{aligned}
\lim_{\substack{(z_1, z_2) \in \overline{S} \\ (z_1, z_2) \to (\overline{z_1}, \overline{z_2})}} w_1^S(z_1, z_2) &= -\frac{\lambda_2 p f'(\overline{z_2})}{\lambda_2 \overline{z_1}} \frac{\lambda_2 \overline{z_1}}{1 + \lambda_2 \overline{z_2} - \lambda_2 \overline{z_1} f'(\overline{z_1})} w(\overline{z_1}, \overline{z_2}) \\
&\quad + f'(\overline{z_2}) \frac{1 + \lambda_2 \overline{z_2}}{1 + \lambda_2 \overline{z_2} - \lambda_2 \overline{z_1} f'(\overline{z_1})} w_2^D(\overline{z_1}, \overline{z_2}) + \frac{1 + \lambda_2 \overline{z_2}}{1 + \lambda_2 \overline{z_2} - \lambda_2 \overline{z_1} f'(\overline{z_1})} w_1^D(\overline{z_1}, \overline{z_2}) \\
&= \frac{f'(\overline{z_2})}{1 + \lambda_2 \overline{z_2} - \lambda_2 \overline{z_1} f'(\overline{z_1})} \left(-\lambda_2 p w(\overline{z_1}, \overline{z_2}) + (1 + \lambda_2 \overline{z_2}) w_2^D(\overline{z_1}, \overline{z_2})\right) \\
&\quad + \frac{1 + \lambda_2 \overline{z_2}}{1 + \lambda_2 \overline{z_2} - \lambda_2 \overline{z_1} f'(\overline{z_1})} w_1^D(\overline{z_1}, \overline{z_2}) \\
&= \frac{f'(\overline{z_2})}{1 + \lambda_2 \overline{z_2} - \lambda_2 \overline{z_1} f'(\overline{z_1})} (-\lambda_2 \overline{z_1} w_1^D(\overline{z_1}, \overline{z_2})) + \frac{1 + \lambda_2 \overline{z_2}}{1 + \lambda_2 \overline{z_2} - \lambda_2 \overline{z_1} f'(\overline{z_1})} w_1^D(\overline{z_1}, \overline{z_2}) \\
&= w_1^D(\overline{z_1}, \overline{z_2}).
\end{aligned}$$

Remark B.16. Under the same assumptions as in Lemma B.15, from (B.27), it follows that

$$\begin{aligned}
w_2^S(z_1, z_2) &= \frac{1}{1 + \lambda_2 z_2} \left(\lambda_2 p w(z_1, z_2) - \lambda_2 z_1 w_1^S(z_1, z_2)\right) \\
&\to \frac{1}{1 + \lambda_2 \overline{z_2}} \left(\lambda_2 p w(\overline{z_1}, \overline{z_2}) - \lambda_2 \overline{z_1} w_1^D(\overline{z_1}, \overline{z_2})\right) = w_2^D(\overline{z_1}, \overline{z_2})
\end{aligned}$$

Lemma B.17. Under the Assumptions B.1, let $(\overline{z}_1, \overline{z}_2) \in \partial S \cap \partial SW$ lie on the boundary of western and southwestern regions, and assume that

$$\mathcal{B}_2(w^D)(z_{1,SW}, z_{2,SW}) = 0, \tag{B.30}$$

$$\mathcal{B}_1(w^D)(z_{1,SW}, z_{2,SW}) = 0, \tag{B.31}$$

Then

$$\lim_{\substack{(z_1,z_2)\in\overline{S}\\(z_1,z_2)\to(\overline{z}_1,\overline{z}_2)}} w_1^S(z_1,z_2) = \lim_{\substack{(z_1,z_2)\in\overline{SW}\\(z_1,z_2)\to(\overline{z}_1,\overline{z}_2)}} w_1^{SW}(z_1,z_2). \tag{B.32}$$

Moreover,

$$\lim_{\substack{(z_1,z_2)\in\overline{D}\\(z_1,z_2)\to(z_{1,SW},z_{2,SW})}} w_i^D(z_1,z_2) = \lim_{\substack{(z_1,z_2)\in\overline{SW}\\(z_1,z_2)\to(z_{1,SW},z_{2,SW})}} w_i^{SW}(z_1,z_2),\ i=1,2, \tag{B.33}$$

which means that w is continuously differentiable at $(z_{1,SW}, z_{2,SW})$.

Proof: We first prove (B.33). Note the from (B.30) and (B.31) we have that

$$w_i^D(z_{1,SW},z_{2,SW}) = \frac{\lambda_i p}{1+\lambda_1 z_{1,SW}+\lambda_2 z_{2,SW}} w(z_{1,SW},z_{2,SW}),\ i=1,2 \tag{B.34}$$

Also using (B.6), we have

$$\begin{aligned}\lim_{\substack{(z_1,z_2)\in\overline{SW}\\(z_1,z_2)\to(\overline{z}_1,\overline{z}_2)}} w_i^{SW}(z_1,z_2) &= \frac{\lambda_i p}{1+\lambda_1\overline{z}_1+\lambda_2\overline{z}_2} w^{SW}(\overline{z}_1,\overline{z}_2) \\ &= \left(\frac{\overline{z}_1}{z_{1,SW}}\right)^p \frac{\lambda_i p}{1+\lambda_1\overline{z}_1+\lambda_2\overline{z}_2} w(z_{1,SW},z_{2,SW})\end{aligned} \tag{B.35}$$

Specifically, if $(\overline{z}_1,\overline{z}_2) = (z_{1,SW}, z_{2,SW})$ then equation (B.33) follows.

To prove (B.32), we note that when $(\overline{z}_1,\overline{z}_2)\in\partial S\cap\partial SW$, we have that $(\widehat{z}_1,\widehat{z}_2) = (z_{1,SW}, z_{2,SW})$. From (B.26) we have

$$\begin{aligned}\lim_{\substack{(z_1,z_2)\in\overline{S}\\(z_1,z_2)\to(\overline{z}_1,\overline{z}_2)}} w_1^S(z_1,z_2) &= \left(\frac{\overline{z}_1}{z_{1,SW}}\right)^p \frac{\lambda_2 z_{1,SW}}{1+\lambda_2\overline{z}_2-\lambda_2\overline{z}_1 f'(z_{1,SW})}\times \\ &\quad \left(-\frac{\lambda_2 p f'(z_{1,SW})}{\lambda_2 z_{1,SW}} w(z_{1,SW},z_{2,SW}) + \frac{1+\lambda_2 z_{2,SW}}{\lambda_2 z_{1,SW}}\left[f'(z_{1,SW})w_2^D + w_1^D\right](z_{1,SW},z_{2,SW})\right) \\ &= \left(\frac{\overline{z}_1}{z_{1,SW}}\right)^p \frac{1+\lambda_2 z_{2,SW}-\lambda_2 z_{1,SW} f'(z_{1,SW})}{1+\lambda_2\overline{z}_2-\lambda_2\overline{z}_1 f'(z_{1,SW})} w_1^D(z_{1,SW},z_{2,SW}) \\ &= \left(\frac{\overline{z}_1}{z_{1,SW}}\right)^p \frac{1+\lambda_2 z_{2,SW}-\lambda_2 z_{1,SW} f'(z_{1,SW})}{1+\lambda_2\overline{z}_2-\lambda_2\overline{z}_1 f'(z_{1,SW})} \frac{\lambda_1 p}{1+\lambda_1 z_{1,SW}+\lambda_2 z_{2,SW}} w^D(z_{1,SW},z_{2,SW}),\end{aligned} \tag{B.36}$$

where we have used (B.30) and (B.34) to get the last two equalities. Finally we see that

$$\frac{1+\lambda_2 z_{2,SW}-\lambda_2 z_{1,SW} f'(z_{1,SW})}{1+\lambda_2\overline{z}_2-\lambda_2\overline{z}_1 f'(z_{1,SW})} = \frac{1+\lambda_1 z_{1,SW}+\lambda_2 z_{2,SW}}{1+\lambda_1\overline{z}_1+\lambda_2\overline{z}_2},$$

because from (B.12) we have that $\frac{1+\lambda_2\overline{z}_2}{1+\lambda_2 z_{2,SW}} = \frac{\overline{z}_1}{z_{1,SW}}$.

□

Remark B.18. Under the same assumptions of Lemma B.17, and similar to Remark B.16, since both $w^S(z_1, z_2)$ and $w^{SW}(z_1, z_2)$ satisfy equation (B.2), it follows that

$$w_2^S(\overline{z}_1, \overline{z}_2) = \frac{1}{1+\lambda_2\overline{z}_2}\left(\lambda_2 p w(\overline{z}_1, \overline{z}_2) - \lambda_2\overline{z}_1 w_1^S(\overline{z}_1, \overline{z}_2)\right),$$
$$w_2^{SW}(\underline{z}_1, \underline{z}_2) = \frac{1}{1+\lambda_2\underline{z}_2}\left(\lambda_2 p w(\underline{z}_1, \underline{z}_2) - \lambda_2\underline{z}_1 w_1^{SW}(\underline{z}_1, \underline{z}_2)\right).$$

Taking the limit, it follows from Lemma B.17 that

$$\lim_{\substack{(\overline{z}_1, \overline{z}_2) \in S \\ (\overline{z}_1, \overline{z}_2) \to (z_1, z_2)}} w_2^S(\overline{z}_1, \overline{z}_2) = \lim_{\substack{(\underline{z}_1, \underline{z}_2) \in SW \\ (\underline{z}_1, \underline{z}_2) \to (z_1, z_2)}} w_2^{SW}(\underline{z}_1, \underline{z}_2).$$

□

We summarize the result of this section so far in the following two theorems.

Theorem B.19. Under the Assumptions B.1, there exists a unique extension w of w^D from $\overline{D}$ to the $\overline{S}$ and the $\overline{SW}$ regions that coincides with w^S and w^{SW} on $\overline{S}$ and $\overline{SW}$ respectively. Moreover, it satisfies $\mathcal{B}_2(w) = 0$ in S and $\mathcal{B}_2(w) = 0 = \mathcal{B}_1(w)$ in SW. It is given by $w^S(z_1, z_2) = \left(\frac{1+\lambda_2 z_2}{1+\lambda_2\widehat{z}_2}\right)^p w^D(\widehat{z}_1, \widehat{z}_2) = \left(\frac{z_1}{\widehat{z}_1}\right)^p w^D(\widehat{z}_1, \widehat{z}_2)$ and $w^{SW}(z_1, z_2) = \left(\frac{1+\lambda_1 z_1+\lambda_2 z_2}{1+\lambda_1 z_{1,SW}+\lambda_2 z_{2,SW}}\right)^p w^D(z_{1,SW}, z_{2,SW})$ in $\overline{S}$ and $\overline{SW}$ respectively. Moreover w is continuous.

In addition, inside the S region we have,

$$\begin{aligned} w_2(z_1, z_2) &\qquad\qquad \text{(B.37)}\\ = \frac{\lambda_2 p}{1+\lambda_2 z_2}\left(\frac{z_1}{\widehat{z}_1}\right)^p w(\widehat{z}_1, \widehat{z}_2) &\\ + \left(\frac{z_1}{\widehat{z}_1}\right)^p \frac{\lambda_2\widehat{z}_1}{1+\lambda_2 z_2 - \lambda_2 z_1 f'(\widehat{z}_1)}\left[\frac{\lambda_2 p f'(\widehat{z}_1)}{1+\lambda_2\widehat{z}_2} w(\widehat{z}_1, \widehat{z}_2) - f'(\widehat{z}_1) w_2^D(\widehat{z}_1, \widehat{z}_2) - w_1^D(\widehat{z}_1, \widehat{z}_2)\right], &\\ w_1(z_1, z_2) = -\left(\frac{z_1}{\widehat{z}_1}\right)^p \frac{1}{1+\lambda_2 z_2 - \lambda_2 z_1 f'(\widehat{z}_1)} &\qquad\qquad \text{(B.38)}\\ \times\left[\lambda_2 p f'(\widehat{z}_1) w(\widehat{z}_1, \widehat{z}_2) - (1+\lambda_2\widehat{z}_2)\left(f'(\widehat{z}_1) w_2^D(\widehat{z}_1, \widehat{z}_2) + w_1^D(\widehat{z}_1, \widehat{z}_2)\right)\right]. & \end{aligned}$$

Moreover, if for all $(z_1, z_2) \in \partial D \cap \partial S$, $\mathcal{B}_2(w^D)(\overline{z}_1, \overline{z}_2) = 0$, then the extended function w has continuous first derivatives across the southern boundary of the D excluding the two endpoints.

If in addition $\mathcal{B}_2(w)(z_{1,SW}, z_{2,SW}) = \mathcal{B}_1(w)(z_{1,SW}, z_{2,SW}) = 0$, then the extended function w has continuous first derivatives at $(z_{1,SW}, z_{2,SW})$

Proof of Theorem B.2: From Proposition B.11 the extended function w is in $C(\mathcal{S}_u)$. From Lemmas B.17 and B.15 and Remarks B.16 and B.18 and their analogues for the other regions, we additionally conclude that $w \in C^1(\mathcal{S}_u)$. □

Corollary B.20. Under the Assumptions B.1, and assuming that for all $(z_1, z_2) \in \partial D \cap \partial S$, $\mathcal{B}_2(w^D)(\overline{z}_1, \overline{z}_2) = 0$, then for any $(z_1, z_2) \in S$

$$w_1(z_1, z_2) = \left(\frac{z_1}{\widehat{z}_1}\right)^{p-1} w_1^D(\widehat{z}_1, \widehat{z}_2), \tag{B.39}$$

$$w_2(z_1, z_2) = \left(\frac{z_1}{\widehat{z}_1}\right)^{p-1} w_2^D(\widehat{z}_1, \widehat{z}_2). \tag{B.40}$$

Proof: All the requirements of Theorem B.2 are satisfied, so we conclude w has continuous first derivatives across the southern boundary of D. Hence, we can calculate the derivatives $w_i(\widehat{z}_1, \widehat{z}_2)$, $i = 1, 2$ by evaluating the (B.25) and (B.26) at $(z_1, z_2) = (\widehat{z}_1, \widehat{z}_2)$. We get

$$\begin{aligned} w_2^S(\widehat{z}_1, \widehat{z}_2) &= \frac{\lambda_2 p}{1 + \lambda_2 \widehat{z}_2} w(\widehat{z}_1, \widehat{z}_2) + \frac{\lambda_2 \widehat{z}_1}{1 + \lambda_2 \widehat{z}_2 - \lambda_2 \widehat{z}_1 f'(\widehat{z}_1)} \\ &\quad \times \left[\frac{\lambda_2 p f'(\widehat{z}_1)}{1 + \lambda_2 \widehat{z}_2} w(\widehat{z}_1, \widehat{z}_2) - f'(\widehat{z}_1) w_2^D(\widehat{z}_1, \widehat{z}_2) - w_1^D(\widehat{z}_1, \widehat{z}_2)\right] \end{aligned} \tag{B.41}$$

$$\begin{aligned} w_1^S(\widehat{z}_1, \widehat{z}_2) &= -\frac{\lambda_2 \widehat{z}_1}{1 + \lambda_2 \widehat{z}_2 - \lambda_2 \widehat{z}_1 f'(\widehat{z}_1)} \\ &\quad \times \left[\frac{\lambda_2 p f'(\widehat{z}_1)}{\lambda_2 \widehat{z}_1} w(\widehat{z}_1, \widehat{z}_2) - \frac{1 + \lambda_2 \widehat{z}_2}{\lambda_2 \widehat{z}_1} \left(f'(\widehat{z}_1) w_2^D(\widehat{z}_1, \widehat{z}_2) + w_1^D(\widehat{z}_1, \widehat{z}_2)\right)\right]. \end{aligned} \tag{B.42}$$

But (B.25) and (B.26) also imply

$$\begin{aligned} w_2^S(z_1, z_2) &= \left(\frac{z_1}{\widehat{z}_1}\right)^p \frac{\lambda_2 p}{1 + \lambda_2 \widehat{z}_2} \frac{1 + \lambda_2 \widehat{z}_2}{1 + \lambda_2 z_2} w(\widehat{z}_1, \widehat{z}_2) + \left(\frac{z_1}{\widehat{z}_1}\right)^p \frac{\lambda_2 \widehat{z}_1}{1 + \lambda_2 \widehat{z}_2 - \lambda_2 \widehat{z}_1 f'(\widehat{z}_1)} \\ &\quad \times \frac{1 + \lambda_2 \widehat{z}_2 - \lambda_2 \widehat{z}_1 f'(\widehat{z}_1)}{1 + \lambda_2 z_2 - \lambda_2 z_1 f'(\widehat{z}_1)} \left[\frac{\lambda_2 p f'(\widehat{z}_1)}{1 + \lambda_2 \widehat{z}_2} w(\widehat{z}_1, \widehat{z}_2) - f'(\widehat{z}_1) w_2^D(\widehat{z}_1, \widehat{z}_2) - w_1^D(\widehat{z}_1, \widehat{z}_2)\right] \end{aligned} \tag{B.43}$$

$$\begin{aligned} w_1^S(z_1, z_2) &= -\left(\frac{z_1}{\widehat{z}_1}\right)^p \frac{\lambda_2 \widehat{z}_1}{1 + \lambda_2 \widehat{z}_2 - \lambda_2 \widehat{z}_1 f'(\widehat{z}_1)} \frac{1 + \lambda_2 \widehat{z}_2 - \lambda_2 \widehat{z}_1 f'(\widehat{z}_1)}{1 + \lambda_2 z_2 - \lambda_2 z_1 f'(\widehat{z}_1)} \\ &\quad \times \left[\frac{\lambda_2 p f'(\widehat{z}_1)}{\lambda_2 \widehat{z}_1} w(\widehat{z}_1, \widehat{z}_2) - \frac{1 + \lambda_2 \widehat{z}_2}{\lambda_2 \widehat{z}_1} \left(f'(\widehat{z}_1) w_2^D(\widehat{z}_1, \widehat{z}_2) + w_1^D(\widehat{z}_1, \widehat{z}_2)\right)\right]. \end{aligned} \tag{B.44}$$

Now use (B.41) in (B.43) together with (B.13) to get (B.40). Similarly, use (B.42) in (B.44) together with (B.13) to get (B.39).

□

B.4 Second derivatives

Differentiating (B.21), (B.20) and (B.20) with respect to z_1, z_2 and z_1, respectively, we get

$$\begin{aligned}\frac{\partial^2 \widehat{z}_1}{\partial z_1^2} &= \frac{\partial}{\partial z_1}\left(\frac{1+\lambda_2\widehat{z}_2}{1+\lambda_2 z_2-\lambda_2 z_1 f'(\widehat{z}_1)}\right) \\ &= \frac{\lambda_2\frac{\partial\widehat{z}_2}{\partial z_1}}{1+\lambda_2 z_2-\lambda_2 z_1 f'(\widehat{z}_1)} - \frac{(1+\lambda_2\widehat{z}_2)(-\lambda_2 f'(\widehat{z}_1)-\lambda_2 z_1 f''(\widehat{z}_1)\frac{\partial\widehat{z}_1}{\partial z_1})}{(1+\lambda_2 z_2-\lambda_2 z_1 f'(\widehat{z}_1))^2} \\ &= \frac{2\lambda_2 f'(\widehat{z}_1)(1+\lambda_2\widehat{z}_2)}{(1+\lambda_2 z_2-\lambda_2 z_1 f'(\widehat{z}_1))^2} + \frac{\lambda_2 z_1 f''(\widehat{z}_1)(1+\lambda_2\widehat{z}_2)^2}{(1+\lambda_2 z_2-\lambda_2 z_1 f'(\widehat{z}_1))^3},\end{aligned} \tag{B.45}$$

$$\begin{aligned}\frac{\partial^2 \widehat{z}_1}{\partial z_2^2} &= \frac{\partial}{\partial z_2}\left(\frac{-\lambda_2\widehat{z}_1}{1+\lambda_2 z_2-\lambda_2 z_1 f'(\widehat{z}_1)}\right) \\ &= \frac{-\lambda_2\frac{\partial\widehat{z}_1}{\partial z_2}}{1+\lambda_2 z_2-\lambda_2 z_1 f'(\widehat{z}_1)} + \frac{\lambda_2\widehat{z}_1(\lambda_2-\lambda_2 z_1 f''(\widehat{z}_1)\frac{\partial\widehat{z}_1}{\partial z_2})}{(1+\lambda_2 z_2-\lambda_2 z_1 f'(\widehat{z}_1))^2} \\ &= \frac{2\lambda_2^2\widehat{z}_1}{(1+\lambda_2 z_2-\lambda_2 z_1 f'(\widehat{z}_1))^2} + \frac{\lambda_2^3 z_1\widehat{z}_1^2 f''(\widehat{z}_1)}{(1+\lambda_2 z_2-\lambda_2 z_1 f'(\widehat{z}_1))^3},\end{aligned} \tag{B.46}$$

$$\begin{aligned}\frac{\partial^2 \widehat{z}_1}{\partial z_2\partial z_1} &= \frac{\partial}{\partial z_1}\left(\frac{-\lambda_2\widehat{z}_1}{1+\lambda_2 z_2-\lambda_2 z_1 f'(\widehat{z}_1)}\right) \\ &= \frac{-\lambda_2\frac{\partial\widehat{z}_1}{\partial z_1}}{1+\lambda_2 z_2-\lambda_2 z_1 f'(\widehat{z}_1)} + \frac{(\lambda_2\widehat{z}_1)(-\lambda_2 f'(\widehat{z}_1)-\lambda_2 z_1 f''(\widehat{z}_1)\frac{\partial\widehat{z}_1}{\partial z_1})}{(1+\lambda_2 z_2-\lambda_2 z_1 f'(\widehat{z}_1))^2} \\ &= -\frac{\lambda_2(1+\lambda_2\widehat{z}_2)+\lambda_2^2\widehat{z}_1 f'(\widehat{z}_1)}{(1+\lambda_2 z_2-\lambda_2 z_1 f'(\widehat{z}_1))^2} - \frac{\lambda_2^2 z_1\widehat{z}_1 f''(\widehat{z}_1)(1+\lambda_2\widehat{z}_2)}{(1+\lambda_2 z_2-\lambda_2 z_1 f'(\widehat{z}_1))^3}.\end{aligned} \tag{B.47}$$

From (B.45), (B.46) and (B.47) it now follows that

$$\begin{aligned}\frac{\partial^2 \widehat{z}_2}{\partial z_1^2} &= \frac{\partial^2}{\partial z_1^2}\left(f(\widehat{z}_1)\right) = f''(\widehat{z}_1)\left(\frac{\partial\widehat{z}_1}{\partial z_1}\right)^2 + f'(\widehat{z}_1)\frac{\partial^2\widehat{z}_1}{\partial z_1^2} \\ &= \frac{2\lambda_2\left(f'(\widehat{z}_1)\right)^2(1+\lambda_2\widehat{z}_2)}{(1+\lambda_2 z_2-\lambda_2 z_1 f'(\widehat{z}_1))^2} + \frac{(1+\lambda_2 z_2)f''(\widehat{z}_1)(1+\lambda_2\widehat{z}_2)^2}{(1+\lambda_2 z_2-\lambda_2 z_1 f'(\widehat{z}_1))^3},\end{aligned} \tag{B.48}$$

$$\begin{aligned}\frac{\partial^2 \widehat{z}_2}{\partial z_2^2} &= \frac{\partial^2}{\partial z_2^2}\left(f(\widehat{z}_1)\right) \\ &= f''(\widehat{z}_1)\left(\frac{\partial\widehat{z}_1}{\partial z_2}\right)^2 + f'(\widehat{z}_1)\frac{\partial^2\widehat{z}_1}{\partial z_2^2} = f''(\widehat{z}_1)\frac{\lambda_2^2\widehat{z}_1^2}{(1+\lambda_2 z_2-\lambda_2 z_1 f'(\widehat{z}_1))^2} \\ &\quad + f'(\widehat{z}_1)\frac{2\lambda_2^2\widehat{z}_1}{(1+\lambda_2 z_2-\lambda_2 z_1 f'(\widehat{z}_1))^2} + \frac{\lambda_2^3 z_1\widehat{z}_1^2 f'(\widehat{z}_1)f''(\widehat{z}_1)}{(1+\lambda_2 z_2-\lambda_2 z_1 f'(\widehat{z}_1))^3} \\ &= \frac{2\lambda_2^2\widehat{z}_1 f'(\widehat{z}_1)}{(1+\lambda_2 z_2-\lambda_2 z_1 f'(\widehat{z}_1))^2} + \frac{\lambda_2^2\widehat{z}_1^2(1+\lambda_2 z_2)f''(\widehat{z}_1)}{(1+\lambda_2 z_2-\lambda_2 z_1 f'(\widehat{z}_1))^3},\end{aligned} \tag{B.49}$$

$$
\begin{aligned}
\frac{\partial^2 \widehat{z}_2}{\partial z_1 \partial z_2} &= \frac{\partial}{\partial z_1}\left(f'(\widehat{z}_1)\frac{\partial \widehat{z}_1}{\partial z_2}\right) = f''(\widehat{z}_1)\frac{\partial \widehat{z}_1}{\partial z_2}\frac{\partial \widehat{z}_1}{\partial z_1} + f'(\widehat{z}_1)\frac{\partial^2 \widehat{z}_1}{\partial z_1 \partial z_2} \qquad \text{(B.50)}\\
&= -f''(\widehat{z}_1)\frac{\lambda_2 \widehat{z}_1(1+\lambda_2\widehat{z}_2)}{(1+\lambda_2 z_2 - \lambda_2 z_1 f'(\widehat{z}_1))^2} - \frac{\lambda_2 f'(\widehat{z}_1)(1+\lambda_2\widehat{z}_2) + \lambda_2^2 \widehat{z}_1 (f'(\widehat{z}_1))^2}{(1+\lambda_2 z_2 - \lambda_2 z_1 f'(\widehat{z}_1))^2}\\
&\quad - \frac{\lambda_2^2 z_1 \widehat{z}_1 f'(\widehat{z}_1) f''(\widehat{z}_1)(1+\lambda_2\widehat{z}_2)}{(1+\lambda_2 z_2 - \lambda_2 z_1 f'(\widehat{z}_1))^3}\\
&= -\frac{\lambda_2 f'(\widehat{z}_1)(1+\lambda_2\widehat{z}_2 + \lambda_2\widehat{z}_1 f'(\widehat{z}_1))}{(1+\lambda_2 z_2 - \lambda_2 z_1 f'(\widehat{z}_1))^2} - \frac{\lambda_2(1+\lambda_2 z_2)(1+\lambda_2\widehat{z}_2)\widehat{z}_1 f''(\widehat{z}_1)}{(1+\lambda_2 z_2 - \lambda_2 z_1 f'(\widehat{z}_1))^3}.
\end{aligned}
$$

Remark B.21. Note that the second derivatives $\frac{\partial^2 \widehat{z}_1}{\partial z_2^2},\ \frac{\partial^2 \widehat{z}_2}{\partial z_2^2} = O(\lambda^2),\ \frac{\partial^2 \widehat{z}_1}{\partial z_1^2},\ \frac{\partial^2 \widehat{z}_1}{\partial z_2 \partial z_1},\ \frac{\partial^2 \widehat{z}_2}{\partial z_1 \partial z_2} = O(\lambda)$, and $\frac{\partial^2 \widehat{z}_2}{\partial z_1^2} = f''(\widehat{z}_1) + O(\lambda)$.

Remark B.22. Note that from (B.12), we have $\frac{z_1}{1+\lambda_2 z_2} = \frac{\widehat{z}_1}{1+\lambda_2 \widehat{z}_2}$. The reader can verify that $z_1^2 \frac{\partial^2 \widehat{z}_i}{\partial z_j^2},\ i,j = 1,2$ and $z_1^2 \frac{\partial^2 \widehat{z}_i}{\partial z_1 \partial z_2},\ i = 1,2$ depends on (z_1, z_2) only through $(\widehat{z}_1, \widehat{z}_2)$.

Differentiating (B.25) and (B.26) we get

$$
\begin{aligned}
w_{11}^S(z_1,z_2) &= \frac{\lambda_2^2 p(p+1)(1+\lambda_2 z_2)^p}{(1+\lambda_2\widehat{z}_2)^{p+2}}\left(\frac{\partial \widehat{z}_2}{\partial z_1}\right)^2 w(\widehat{z}_1,\widehat{z}_2) \qquad \text{(B.51)}\\
&\quad - \frac{\lambda_2 p(1+\lambda_2 z_2)^p}{(1+\lambda_2\widehat{z}_2)^{p+1}}\frac{\partial^2 \widehat{z}_2}{\partial z_1^2} w(\widehat{z}_1,\widehat{z}_2)\\
&\quad - \frac{2\lambda_2 p(1+\lambda_2 z_2)^p}{(1+\lambda_2\widehat{z}_2)^{p+1}}\frac{\partial \widehat{z}_2}{\partial z_1}\left(w_2^D(\widehat{z}_1,\widehat{z}_2)\frac{\partial \widehat{z}_2}{\partial z_1} + w_1^D(\widehat{z}_1,\widehat{z}_2)\frac{\partial \widehat{z}_1}{\partial z_1}\right)\\
&\quad + \left(\frac{1+\lambda_2 z_2}{1+\lambda_2\widehat{z}_2}\right)^p \left(w_2^D(\widehat{z}_1,\widehat{z}_2)\frac{\partial^2 \widehat{z}_2}{\partial z_1^2} + w_1^D(\widehat{z}_1,\widehat{z}_2)\frac{\partial^2 \widehat{z}_1}{\partial z_1^2}\right)\\
&\quad + \left(\frac{1+\lambda_2 z_2}{1+\lambda_2\widehat{z}_2}\right)^p\\
&\qquad \times \left(\left(\frac{\partial \widehat{z}_2}{\partial z_1}\right)^2 w_{22}^D(\widehat{z}_1,\widehat{z}_2) + 2\frac{\partial \widehat{z}_2}{\partial z_1}\frac{\partial \widehat{z}_1}{\partial z_1} w_{12}^D(\widehat{z}_1,\widehat{z}_2) + \left(\frac{\partial \widehat{z}_1}{\partial z_1}\right)^2 w_{11}^D(\widehat{z}_1,\widehat{z}_2)\right),
\end{aligned}
$$

$$
\begin{aligned}
w^S_{12}(z_1,z_2) &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(B.52)}\\
& \frac{\partial}{\partial z_2}\left[\left(\frac{z_1}{\widehat{z}_1}\right)^p\left(\frac{-\lambda_2 p}{1+\lambda_2\widehat{z}_2}\frac{\partial\widehat{z}_2}{\partial z_1}w(\widehat{z}_1,\widehat{z}_2)+\frac{\partial\widehat{z}_2}{\partial z_1}w^D_2(\widehat{z}_1,\widehat{z}_2)+\frac{\partial\widehat{z}_1}{\partial z_1}w^D_1(\widehat{z}_1,\widehat{z}_2)\right)\right]\\
&= -p\frac{z_1^p}{\widehat{z}_1^{p+1}}\left(\frac{\partial\widehat{z}_1}{\partial z_2}\right)\left[\frac{-\lambda_2 p}{1+\lambda_2\widehat{z}_2}\frac{\partial\widehat{z}_2}{\partial z_1}w(\widehat{z}_1,\widehat{z}_2)+\frac{\partial\widehat{z}_2}{\partial z_1}w^D_2(\widehat{z}_1,\widehat{z}_2)+\frac{\partial\widehat{z}_1}{\partial z_1}w^D_1(\widehat{z}_1,\widehat{z}_2)\right]\\
&\quad+\left(\frac{z_1}{\widehat{z}_1}\right)^p\left[\frac{\lambda_2^2 p}{(1+\lambda_2\widehat{z}_2)^2}\frac{\partial\widehat{z}_2}{\partial z_2}\frac{\partial\widehat{z}_2}{\partial z_1}w(\widehat{z}_1,\widehat{z}_2)-\frac{\lambda_2 p}{1+\lambda_2\widehat{z}_2}\frac{\partial^2\widehat{z}_2}{\partial z_2\partial z_1}w(\widehat{z}_1,\widehat{z}_2)\right.\\
&\quad-\frac{\lambda_2 p}{1+\lambda_2\widehat{z}_2}\frac{\partial\widehat{z}_2}{\partial z_1}\left(w^D_2(\widehat{z}_1,\widehat{z}_2)\frac{\partial\widehat{z}_2}{\partial z_2}+w^D_1(\widehat{z}_1,\widehat{z}_2)\frac{\partial\widehat{z}_1}{\partial z_2}\right)\\
&\quad+w^D_2(\widehat{z}_1,\widehat{z}_2)\frac{\partial^2\widehat{z}_2}{\partial z_2\partial z_1}+w^D_1(\widehat{z}_1,\widehat{z}_2)\frac{\partial^2\widehat{z}_1}{\partial z_2\partial z_1}+\frac{\partial\widehat{z}_2}{\partial z_1}\frac{\partial\widehat{z}_2}{\partial z_2}w^D_{22}(\widehat{z}_1,\widehat{z}_2)\\
&\quad\left.+\frac{\partial\widehat{z}_2}{\partial z_1}\frac{\partial\widehat{z}_1}{\partial z_2}w^D_{12}(\widehat{z}_1,\widehat{z}_2)+\frac{\partial\widehat{z}_2}{\partial z_2}\frac{\partial\widehat{z}_1}{\partial z_1}w^D_{12}(\widehat{z}_1,\widehat{z}_2)+\frac{\partial\widehat{z}_1}{\partial z_2}\frac{\partial\widehat{z}_1}{\partial z_1}w^D_{11}(\widehat{z}_1,\widehat{z}_2)\right],
\end{aligned}
$$

$$
\begin{aligned}
w^S_{22}(z_1,z_2) &= \frac{\partial}{\partial z_2}\left(\frac{z_1}{\widehat{z}_1}\right)^p\left[\frac{\lambda_2 p}{1+\lambda_2 z_2}w(\widehat{z}_1,\widehat{z}_2)-\frac{\lambda_2 p}{1+\lambda_2\widehat{z}_2}w(\widehat{z}_1,\widehat{z}_2)\frac{\partial\widehat{z}_2}{\partial z_2}\right. \qquad \text{(B.53)}\\
&\qquad\left.+\frac{\partial\widehat{z}_2}{\partial z_2}w^D_2(\widehat{z}_1,\widehat{z}_2)+\frac{\partial\widehat{z}_1}{\partial z_2}w^D_1(\widehat{z}_1,\widehat{z}_2)\right]\\
&= -p\frac{z_1^p}{\widehat{z}_1^{p+1}}\left(\frac{\partial\widehat{z}_1}{\partial z_2}\right)\left[\frac{\lambda_2 p}{1+\lambda_2 z_2}w(\widehat{z}_1,\widehat{z}_2)-\frac{\lambda_2 p}{1+\lambda_2\widehat{z}_2}w(\widehat{z}_1,\widehat{z}_2)\frac{\partial\widehat{z}_2}{\partial z_2}\right.\\
&\qquad\left.+\frac{\partial\widehat{z}_2}{\partial z_2}w^D_2(\widehat{z}_1,\widehat{z}_2)+\frac{\partial\widehat{z}_1}{\partial z_2}w^D_1(\widehat{z}_1,\widehat{z}_2)\right]\\
&\quad+\left(\frac{z_1}{\widehat{z}_1}\right)^p\left[\left(\frac{\partial\widehat{z}_2}{\partial z_2}w^D_2(\widehat{z}_1,\widehat{z}_2)+\frac{\partial\widehat{z}_1}{\partial z_2}w^D_1(\widehat{z}_1,\widehat{z}_2)\right)\left(\frac{\lambda_2 p}{1+\lambda_2 z_2}-\frac{\partial\widehat{z}_2}{\partial z_2}\frac{\lambda_2 p}{1+\lambda_2\widehat{z}_2}\right)\right.\\
&\quad+\lambda_2 p w(\widehat{z}_1,\widehat{z}_2)\left(\frac{-\lambda_2}{(1+\lambda_2 z_2)^2}-\frac{\frac{\partial^2\widehat{z}_2}{\partial z_2^2}}{1+\lambda_2\widehat{z}_2}+\left(\frac{\partial\widehat{z}_2}{\partial z_2}\right)^2\frac{\lambda_2}{(1+\lambda_2\widehat{z}_2)^2}\right)\\
&\quad+w^D_2(\widehat{z}_1,\widehat{z}_2)\frac{\partial^2\widehat{z}_2}{\partial z_2^2}+w^D_1(\widehat{z}_1,\widehat{z}_2)\frac{\partial^2\widehat{z}_1}{\partial z_2^2}\\
&\quad\left.+\left(\frac{\partial\widehat{z}_2}{\partial z_2}\right)^2 w^D_{22}(\widehat{z}_1,\widehat{z}_2)+2\frac{\partial\widehat{z}_2}{\partial z_2}\frac{\partial\widehat{z}_1}{\partial z_2}w^D_{12}(\widehat{z}_1,\widehat{z}_2)+\left(\frac{\partial\widehat{z}_1}{\partial z_2}\right)^2 w^D_{11}(\widehat{z}_1,\widehat{z}_2)\right].
\end{aligned}
$$

Remark B.23. If $w^D_i(\widehat{z}_1,\widehat{z}_2)=O(\lambda)$, $w^D_{ij}(\widehat{z}_1,\widehat{z}_2)=O\left(\lambda^{\frac{2}{3}}\right)$, and $f'(\widehat{z}_1)=O\left(\lambda^{\frac{1}{3}}\right)$, then using Remarks B.12 and B.21 it follows that for (z_1,z_2) in a compact set in $\mathcal{S}_u$

$$
\begin{aligned}
w^S_{11}(z_1,z_2) &= \left(f'(\widehat{z}_1)\right)^2 w^D_{22}(\widehat{z}_1,\widehat{z}_2)+2f'(\widehat{z}_1)w^D_{12}(\widehat{z}_1,\widehat{z}_2)+w^D_{11}(\widehat{z}_1,\widehat{z}_2)+O(\lambda)\\
&= w^D_{11}(\widehat{z}_1,\widehat{z}_2)+O(\lambda), \qquad \text{(B.54)}
\end{aligned}
$$

$$w^S_{12}(z_1,z_2)=O\left(\lambda^{\frac{5}{3}}\right), \qquad \text{(B.55)}$$

$$w^S_{22}(z_1,z_2)=O\left(\lambda^2\right). \qquad \text{(B.56)}$$

Lemma B.24. Assume that $w^D(\widehat{z}_1,\widehat{z}_2) = \frac{A^{p-1}}{p} + O(\lambda^{\frac{2}{3}})$, $w_i^D(\widehat{z}_1,\widehat{z}_2) = O(\lambda)$, and $w_{ij}^D(\widehat{z}_1,\widehat{z}_2) = O\left(\lambda^{\frac{2}{3}}\right)$. Then $\frac{1}{\lambda_2^2 z_1^{p-2}} w_{22}^S(z_1,z_2)$ depends on (z_1,z_2) only through $(\widehat{z}_1,\widehat{z}_2)$, and

$$\frac{1}{\lambda_2^2 z_1^{p-2}} w_{22}^S(z_1,z_2) = -\frac{(1-p)A^{p-1}}{\widehat{z}_1^{p-2}} + O\left(\lambda^{\frac{2}{3}}\right). \tag{B.57}$$

Proof: Indeed, from (B.53) and (B.22) we see that,

$$\begin{aligned}
&\frac{1}{z_1^{p-2}} w_{22}^S(z_1,z_2) = -\frac{p}{\widehat{z}_1^{p+1}}\left(z_1\frac{\partial \widehat{z}_1}{\partial z_2}\right)\frac{\lambda_2 p z_1}{1+\lambda_2 z_2} w(\widehat{z}_1,\widehat{z}_2) \\
&+\frac{p}{\widehat{z}_1^{p+1}}\left(z_1\frac{\partial \widehat{z}_1}{\partial z_2}\right)^2\left[\frac{\lambda_2 p}{1+\lambda_2\widehat{z}_2} w(\widehat{z}_1,\widehat{z}_2) f'(\widehat{z}_1) - f'(\widehat{z}_1) w_2^D(\widehat{z}_1,\widehat{z}_2) - w_1^D(\widehat{z}_1,\widehat{z}_2)\right] \\
&+\left(\frac{1}{\widehat{z}_1^p}\right)\left[\lambda_2 p\left(z_1\frac{\partial \widehat{z}_1}{\partial z_2}\right)\left(f'(\widehat{z}_1) w_2^D(\widehat{z}_1,\widehat{z}_2) + w_1^D(\widehat{z}_1,\widehat{z}_2)\right)\right. \\
&\qquad \times\left(\frac{z_1}{1+\lambda_2 z_2} - \left(z_1\frac{\partial \widehat{z}_2}{\partial z_2}\right)\frac{1}{1+\lambda_2\widehat{z}_2}\right) \\
&\qquad +\lambda_2 p w^+(\widehat{z}_1,\widehat{z}_2)\left(\frac{-\lambda_2 z_1^2}{(1+\lambda_2 z_2)^2} - \frac{z_1^2\frac{\partial^2 \widehat{z}_2}{\partial z_2^2}}{1+\lambda_2\widehat{z}_2} + \left(z_1\frac{\partial \widehat{z}_2}{\partial z_2}\right)^2\frac{\lambda_2}{(1+\lambda_2\widehat{z}_2)^2}\right) \\
&\qquad +w_2^D(\widehat{z}_1,\widehat{z}_2) z_1^2\frac{\partial^2 \widehat{z}_2}{\partial z_2^2} + w_1^D(\widehat{z}_1,\widehat{z}_2) z_1^2\frac{\partial^2 \widehat{z}_1}{\partial z_2^2} \\
&\qquad \left.+\left(z_1\frac{\partial \widehat{z}_2}{\partial z_2}\right)^2 w_{22}^D(\widehat{z}_1,\widehat{z}_2) + 2z_1^2\frac{\partial \widehat{z}_2}{\partial z_2}\frac{\partial \widehat{z}_1}{\partial z_2} w_{12}^D(\widehat{z}_1,\widehat{z}_2) + \left(z_1\frac{\partial \widehat{z}_1}{\partial z_2}\right)^2 w_{11}^D(\widehat{z}_1,\widehat{z}_2)\right]
\end{aligned} \tag{B.58}$$

From Remarks B.13 and B.22 the right hand side of (B.58) depends on (z_1,z_2) only through $(\widehat{z}_1,\widehat{z}_2)$. From (B.58) we see that

$$\begin{aligned}
&\frac{1}{\lambda_2^2 z_1^{p-2}} w_{22}^S(z_1,z_2) \\
&= \left[\frac{p}{\widehat{z}_1^{p+1}}\frac{\widehat{z}_1^2}{1+\lambda_2\widehat{z}_2-\lambda_2\widehat{z}_1 f'(\widehat{z}_1)}\frac{p\widehat{z}_1}{1+\lambda_2\widehat{z}_2} - \frac{1}{\widehat{z}_1^p}\frac{p\widehat{z}_1^2}{(1+\lambda_2\widehat{z}_2)^2}\right] w(\widehat{z}_1,\widehat{z}_2) + O\left(\lambda^{\frac{2}{3}}\right) \\
&= -\frac{p(1-p)}{\widehat{z}_1^{p-2}} w(\widehat{z}_1,\widehat{z}_2) + O\left(\lambda^{\frac{2}{3}}\right) = -\frac{(1-p)A^{p-1}}{\widehat{z}_1^{p-2}} + O\left(\lambda^{\frac{2}{3}}\right).
\end{aligned}$$

□

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