Problem 1: Write a MATLAB function that implements the Chinese Remainder Theorem result. The input is two vectors, $\vec{n}$ and $\vec{b}$, of the same length. The vector $\vec{n}$ consists of pairwise relatively prime positive integers and the vector $\vec{b}$ consists of integers. The function outputs $x$, the unique nonnegative integer less than the product of the entries of $\vec{n}$ such that $x$ is congruent to each respective entry of $\vec{b}$ modulo the respective entry of $\vec{n}$. Be sure to provide several nontrivial examples to illustrate that your code is working properly; in particular, the length of $\vec{n}$ and $\vec{b}$ in your examples should be at least 5.

Problem 2:

a) Write a MATLAB function that estimates the probability $\varrho$ that two independently selected integers are relatively prime. The input consists of positive integers $M$ and $N$; your code performs the following experiment $N$ times. It randomly selects two integers between 1 and $M$ (each possibility equally likely) and tests if these two integers are relatively prime. If $n$ times out of the $N$ experiments the two random integers were relatively prime, then the desired probability estimate of $\varrho$ is $\frac{n}{N}$.

b) Test your code when $M = 10,000,000$ and $N = 10,000,000$ to estimate $\varrho$.

c) Argue that the probability is $1 - \frac{1}{2^2}$ that two independently selected integers don’t have common divisor 2. Argue that the probability is $1 - \frac{1}{3^2}$ that two independently selected integers don’t have common divisor 3.

d) Argue that $\varrho = \prod_{p \in P} (1 - \frac{1}{p^2})$, where $P$ denotes the set of all prime numbers. (You may assume the truth of a certain plausible independence.)

e) Compute $\prod (1 - \frac{1}{p^2})$ over all primes $p$ less than 10000 and compare your answer to part b. (Hint: There are 1229 such prime numbers less than 10000, and MATLAB will give you the list with the command `primes(10000)`.)

Problem 3:

a) The Reimann zeta function is defined as $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. Use MATLAB to compute $\sum_{n=1}^{10,000,000} \frac{1}{n^2}$, which will be an approximation of $\zeta(2)$. (Note: The so-called “Basel Problem” solved in the year 1735 by Euler was finding the exact value $\zeta(2) = \frac{\pi^2}{6} = 1.644934...$.)

b) Prove that for any real number $s > 1$, it holds that $\zeta(s) = \prod_{p \in P} (1 - \frac{1}{p^s})^{-1}$, where $P$ denotes the set of all prime numbers. (Hint: Recall from calculus the geometric series; that is, $\sum_{i=0}^{\infty} r^i = (1 - r)^{-1}$ for all $r$ such that $|r| < 1$.)

c) Now show that $\varrho = \frac{1}{\zeta(2)}$, where $\varrho$ is from Problem 2, and give an exact value of $\varrho$ by using Euler’s solution to the Basel Problem that $\zeta(2) = \frac{\pi^2}{6} = 1.644934...$.