Important Instructions: You may discuss this homework with students currently in the class, with the TAs, and with me, up until the time that you do your write-up; but the solutions and code that you submit should be entirely your own. At no time may you consult any existing written solutions; this would be in violation of the homework rules and, in addition, would be plagiarism if sources are not cited.

Problem 1: If \( m \) and \( n \) are integers not both zero, we define their least common multiple \( \text{lcm}(m, n) \) to be the smallest positive integer \( z \) such that \( m|z \) and \( n|z \). Describe how to efficiently compute \( \text{lcm}(m, n) \), and explain why this works and why this is efficient. (Hint: Since there is no known efficient algorithm for factoring integers, there isn’t a known efficient algorithm for computing prime decompositions \( n = \prod p_i^{a_i} \) and \( m = \prod p_i^{b_i} \). Nonetheless, such decompositions exist. Consider a formula for \( \text{lcm}(m, n) \) expressed through the prime decompositions of \( m \) and \( n \); relate this to a formula for \( \gcd(m, n) \) and for \( mn \).)

Problem 2: Suppose \( c \) and \( d \) are positive integers. Show that if \( c^{1/d} \) is not an integer then it is irrational, i.e. it can’t be expressed as \( c^{1/d} = \frac{m}{n} \) for any integers \( m \) and \( n \). (For example, \( \sqrt{2} \) is irrational.) Hint: First characterize when an integer \( z \) is the \( d \)th power of some integer, using its prime decomposition \( z = \prod p_i^{a_i} \).

Problem 3: A continued fraction expansion of a number \( x \) is an expression of \( x \) as

\[
x = \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 + \ddots + \cfrac{1}{q_j + \cdots}}}},
\]

where \( j \) is a positive integer, and \( q_1, q_2, \ldots, q_j \) are positive integers. (Actually, there is a \( q_0 \) which I omitted for simplicity.) For example, the continued fraction expansion for \( \frac{1002}{2501} \) is

\[
\frac{1002}{2501} = \frac{1}{2 + \frac{1}{6 + \frac{1}{8}}}.\]

Explain and justify how to use the Euclid Algorithm to find a continued fraction expansion. (Hint: If \( a, b, q, r \) are as given in the statement of the Division Lemma, simplify \( \frac{1}{q + \frac{r}{b}} \).) Use your method to compute the continued fraction expansion of \( \frac{1337}{3501} \).