Homework 7, 550.362 Optimization, Spring 2016

You may discuss ideas for the homework with other students in the class, but when it comes time to write up your solutions you must do it alone, and your write-up must exactly reflect your understanding. You may not consult other textbooks, online resources, nor any existing solutions.

Problem 1: Prove that for every graph $G = (V,E)$ it holds that $\alpha(G) + \beta(G) = |V|$.

Hint: Suppose that $S \subseteq V$ is an independent set, what can you say about its complement $S^C$?

Solution: Following the hint, for any $S \subseteq V$, $S$ is an independent set if and only if there is no edge $e \in E$ with both endpoints in $S$, if and only if every edge $e \in E$ has at least one endpoint in $S^C$, ie $S^C$ is a vertex cover. Now, consider a maximum cardinality independent set $S$ and a minimum cardinality vertex cover $T$; by the above, $S^C$ is a vertex cover and $T^C$ is an independent set. Thus, $\beta(G) \leq |S^C| = |V| - |S| = |V| - \alpha(G)$ and $\alpha(G) \geq |T^C| = |V| - |T| = |V| - \beta(G)$, which respectively imply $\alpha(G) + \beta(G) \leq |V|$ and $\alpha(G) + \beta(G) \geq |V|$, hence $\alpha(G) + \beta(G) = |V|$.

Problem 2: Suppose that $G = (V,E)$ is a graph with $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$, and $A$ is the incidence matrix of $G$, as defined in lecture (meaning that for all $i, j$ it holds that $A_{i,j}$ is 1 or 0 according as edge $e_i$ has vertex $v_j$ as an endpoint, or not). Consider the integer program

(IP) \begin{align*}
\min & \quad \vec{1}_m^T x \\
\text{s.t.} & \quad A^T x \geq \vec{1}_n \\
& \quad x \in \{0, 1\}^m.
\end{align*}

Give a graph theoretic interpretation for IP and its dual integer program. (Be sure to explain very clearly how your interpretations comes from the respective programs.)

Solution: The entries of vector $x$ are 1 or 0 according as the respective edge of $E$ are in, say, subset $F \subseteq E$. For all $i$, $(A^T x)_i$ is thus the number of edges of $F$ that have vertex $v_i$ as an endpoint. In particular, $A^T x \geq \vec{1}_n$ precisely when $F$ is an edge cover. The objective function $\vec{1}_m^T x = |F|$, so the IP is precisely the min edge cover problem. The dual IP is

(DIP) \begin{align*}
\max & \quad \vec{1}_n^T y \\
\text{s.t.} & \quad A y \leq \vec{1}_m \\
& \quad y \in \{0, 1\}^n.
\end{align*}

The entries of vector $y$ are 1 or 0 according as the respective vertices of $V$ are in, say, subset $S \subseteq V$. For all $i$, $(Ay)_i$ is thus the number of endpoints $e_i$ has in $S$. In particular, $Ay \leq \vec{1}_m$ precisely when $S$ is an independent set. The objective function $\vec{1}_n^T y = |S|$, so the DIP is precisely the max independent set problem.
Problem 3: Consider the weak duality from the integer program and its dual in Problem 2. Express this weak duality in the language of graphs and combinatorially prove this weak duality.

Solution: The weak duality states that, for all graphs $G$, $\alpha(G) \leq \beta'(G)$. To combinatorially prove this weak duality, suppose that $G = (V, E)$ is any graph, and let $S \subseteq V$ be a maximum cardinality independent set of $G$, and $F \subseteq E$ a minimum cardinality edge cover of $G$. Each vertex in $S$ is an endpoint of an edge from $F$ (since $F$ is an edge cover) and this edge cannot have any other vertex of $S$ as an endpoint (since $S$ is an independent set), hence $\alpha(G) = |S| \leq |F| = \beta'(G)$.

Problem 4: Give an example of a duality gap for the integer program and its dual in Problem 2.

Solution: Consider the graph $G$ which is just a 3-cycle. Clearly $\alpha(G) = 1$ since $G$ is a clique, however $\beta'(G) = 2$ since no single edge can cover all three vertices. So here $\alpha(G) < \beta'(G)$.

Problem 5: Give a very wide class of graphs for which strong duality holds in the integer program and its dual in Problem 2, and prove your assertion.

Solution: If $G$ is a bipartite graph with no isolated vertices then $\alpha(G) = \beta'(G)$.

Proof: We have $\alpha(G) = |V| - \beta(G)$ by Problem 1, $\beta(G) = \alpha'(G)$ by Koenig-Egervary Theorem, and $\beta'(G) = |V| - \alpha'(G)$ by Gallai’s Theorem; so $\alpha(G) = |V| - \beta(G) = |V| - \alpha'(G) = \beta'(G)$.

Problem 6: For graph $G = (V, E)$ with $V = \{v_1, v_2, \ldots, v_n\}$, say that $I_1, I_2, \ldots, I_K$ are all of the independent sets in $G$. (It is possible that $K$ is very large. Just for example, if $E$ was empty then every subset of $V$ would be an independent set, and $K$ would be $2^n$.) Define for $G$ the independence incidence matrix $A \in \{0, 1\}^{n \times K}$ such that, for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, K$, we have $A_{i,j}$ is 1 or 0, according as $v_i \in I_j$ or $v_i \not\in I_j$. Consider the integer program

(IP) $\min \vec{1}^T_k x$

s.t. $Ax \geq \vec{1}_n$

$x \in \{0, 1\}^K$.

Give a graph theoretic interpretation for IP and its dual integer program. (Be sure to explain very clearly how your interpretations comes from the respective programs.)

Solution: The entries of vector $x$ are 1 or 0 according as the respective independent set is to be in some family of independent sets, call the family $I$. For all $i$, $(Ax)_i$ is thus the number of
independent sets of $\mathcal{I}$ that vertex $v_i$ is a member of. Particularly, $Ax \geq \mathbf{1}_n$ precisely every vertex is in at least one independent set of $\mathcal{I}$, ie that $\mathcal{I}$ are the color classes for a proper vertex coloring of $G$. (If a vertex is on more than one set from $\mathcal{I}$ then flip a coin to decide among them which color class the vertex will belong.) The objective function $\mathbf{1}_K^T x = |\mathcal{I}|$ is the number of color classes, so the IP is precisely the problem of finding the chromatic number of $G$. The dual IP is

$$\text{(DIP)} \quad \max \mathbf{1}_n^T y$$

$$\text{s.t.} \quad A^T y \leq \mathbf{1}_K$$

$$y \in \{0, 1\}^n.$$ 

The entries of vector $y$ are 1 or 0 according as the respective vertices of $V$ are in, say, subset $S \subseteq V$. For all $i$, $(A^T y)_i$ is thus $|S \cap I_i|$. In particular, $A^T y \leq \mathbf{1}_K$ precisely when $S$ is a clique. (Indeed, if $S$ is a clique then it can intersect any independent set in at most one vertex and, conversely, if $S$ is a non-clique then it has two nonadjacent vertices, and these two vertices form an independent set that intersects $S$ in more than one vertex.) The objective function $\mathbf{1}_n^T y = |S|$, so the DIP is precisely the clique number problem.

**Problem 7:** Consider the weak duality from the integer program and its dual in Problem 6. Express this weak duality in the language of graphs and combinatorially prove this weak duality.

**Solution:** The weak duality states that, for all graphs $G$, $\omega(G) \leq \chi(G)$. To combinatorially prove this weak duality, suppose that $G = (V, E)$ is any graph, and let $S \subseteq V$ be a maximum cardinality clique in $G$. Each vertex in $S$ needs a distinct color, since they are pairwise adjacent, thus the number of colors needed to properly vertex color $G$ is at least $|S|$, ie we have $\omega(G) \leq \chi(G)$.

**Problem 8:** Give an example of a duality gap for the integer program and its dual in Problem 6. Hint: You can find an example with a graph on 5 vertices.

**Solution:** Consider the graph $G$ which is just a 5-cycle. Clearly $\omega(G) = 2$ since $G$ has no triangles, however $\chi(G) = 3$ since two colors alternating around the odd cycle would result in adjacent vertices with the same color. So here $\omega(G) < \chi(G)$.

**Problem 9:** There are radio stations $r_1, r_2, \ldots, r_n$ spread out across the country; for every pair that are within 800 miles, their broadcast frequencies must be different of else their signals will interfere with each other. How few frequencies are required so that every radio station can be assigned a frequency without interference? Just express this problem as a graph theoretic optimization problem.
Solution: Consider the conflict graph \( G = (V, E) \) whose vertices are \( V = \{r_1, r_2, \ldots, r_n\} \), and for every pair of distinct \( r_i, r_j \in V \) we have \( \{r_i, r_j\} \in E \) precisely when these two radio stations are within 800 miles of each other. Then assigning frequencies to the stations is precisely properly coloring the vertices, since different colors (frequencies) are required for adjacent vertices (radio stations). Thus, the minimum number of frequencies needed is the chromatic number of \( G \).

Problem 10: Suppose the cargo boats that dock from time to time in a particular port are \( c_1, c_2, \ldots, c_n \). For each pair of cargo ships, if they are in the port at the same time then they need different docks. How few docks are needed to accommodate all of the cargo ships? Just express this problem as a graph theoretic optimization problem.

Solution: Consider the conflict graph \( G = (V, E) \) whose vertices are \( V = \{c_1, c_2, \ldots, c_n\} \), and for every pair of distinct \( c_i, c_j \in V \) we have \( \{c_i, c_j\} \in E \) precisely when these two cargo ships are in the port at the same time. Then assigning docks to the cargo ships is precisely properly coloring the vertices, since different colors (docks) are required for adjacent vertices (cargo ships). Thus, the minimum number of docks needed is the chromatic number of \( G \).