You may discuss ideas for the homework with other students in the class, but when it comes
time to write up your solutions you must do it alone, and your write-up must exactly reflect your
understanding. You may not consult other textbooks, online resources, nor any existing solutions.

Problem 1: Prove that for every graph \( G = (V,E) \) it holds that \( \alpha(G) + \beta(G) = |V| \).
Hint: Suppose that \( S \subseteq V \) is an independent set, what can you say about its complement \( S^C \)?

Problem 2: Suppose that \( G = (V,E) \) is a graph with \( V = \{v_1, v_2, \ldots, v_n\} \) and \( E = \{e_1, e_2, \ldots, e_m\} \),
and \( A \) is the incidence matrix of \( G \), as defined in lecture (meaning that for all \( i, j \) it holds that \( A_{i,j} \) is 1 or 0 according as edge \( e_i \) has vertex \( v_j \) as an endpoint, or not). Consider the integer program
(IP) \[
\begin{array}{l}
\text{min } \mathbb{I}_m^T x \\
\text{s.t. } A^T x \geq \mathbb{I}_n \\
x \in \{0,1\}^m.
\end{array}
\]
Give a graph theoretic interpretation for IP and its dual integer program. (Be sure to explain
very clearly how your interpretations comes from the respective programs.)

Problem 3: Consider the weak duality from the integer program and its dual in Problem 2.
Express this weak duality in the language of graphs and combinatorially prove this weak duality.

Problem 4: Give an example of a duality gap for the integer program and its dual in Problem 2.

Problem 5: Give a very wide class of graphs for which strong duality holds in the integer pro-
gram and its dual in Problem 2, and prove your assertion.

Problem 6: For graph \( G = (V,E) \) with \( V = \{v_1, v_2, \ldots, v_n\} \), say that \( I_1, I_2, \ldots, I_K \) are all of
the independent sets in \( G \). (It is possible that \( K \) is very large. Just for example, if \( E \) was empty
then every subset of \( V \) would be an independent set, and \( K \) would be \( 2^n \).) Define for \( G \) the in-
dependence incidence matrix \( A \in \{0,1\}^{n \times K} \) such that, for all \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, K \),
we have \( A_{i,j} \) is 1 or 0, according as \( v_i \in I_j \) or \( v_i \notin I_j \). Consider the integer program
(IP) \[
\begin{array}{l}
\text{min } \mathbb{I}_K^T x \\
\text{s.t. } Ax \geq \mathbb{I}_n \\
x \in \{0,1\}^K.
\end{array}
\]
Give a graph theoretic interpretation for IP and its dual integer program. (Be sure to explain
very clearly how your interpretations comes from the respective programs.)
**Problem 7**: Consider the weak duality from the integer program and its dual in Problem 6. Express this weak duality in the language of graphs and combinatorially prove this weak duality.

**Problem 8**: Give an example of a duality gap for the integer program and its dual in Problem 6. Hint: You can find an example with a graph on 5 vertices.

**Problem 9**: There are radio stations $r_1, r_2, \ldots, r_n$ spread out across the country; for every pair that are within 800 miles, their broadcast frequencies must be different or else their signals will interfere with each other. How few frequencies are required so that every radio station can be assigned a frequency without interference? Just express this problem as a graph theoretic optimization problem.

**Problem 10**: Suppose the cargo boats that dock from time to time in a particular port are $c_1, c_2, \ldots, c_n$. For each pair of cargo ships, if they are in the port at the same time then they need different docks. How few docks are needed to accommodate all of the cargo ships? Just express this problem as a graph theoretic optimization problem.