Homework 6, 550.362 Optimization, Spring 2016

**Important Instructions:** You may discuss this homework with students currently in the class, with the TAs, and with me, up until the time that you do your write-up. At no time may you consult any existing written solutions; this would be in violation of the homework rules and, in addition, would be plagiarism if sources are not cited.

**Problem 1:** (Morris 2.1.1) Find all saddle points of

\[
M = \begin{bmatrix}
2 & -1 & 0 & 0 & 1 \\
0 & 0 & 1 & 2 & 1 \\
1 & -2 & 1 & 0 & 2 \\
-1 & 0 & -1 & 1 & 1
\end{bmatrix}
\]

**Solution:** The only saddle point is \((i^*, j^*) = (2, 2)\), which has value 0.

**Problem 2:** (Morris 2.1.2) Find all saddle points of

\[
M = \begin{bmatrix}
1 & -1 & 2 & 2 & 0 \\
-2 & 0 & 1 & 0 & 2 \\
-1 & 0 & -1 & -1 & 0 \\
1 & 1 & 1 & 2 & 1 \\
1 & -1 & 0 & -1 & 1
\end{bmatrix}
\]

**Solution:** There are two saddle points; \((4, 1)\) and \((4, 2)\), each with value 1.

**Problem 3:** (Morris 2.1.3) For which values of \(z\) is there a saddle point for

\[
M = \begin{bmatrix}
-2 & z \\
z & 1
\end{bmatrix}
\]

**Solution:** We have \((1, 1)\) is a saddle point when \(z = -2\), and \((2, 2)\) is a saddle point when \(z = 1\), and \((2, 1)\) is a saddle point when \(-2 \leq z \leq 1\), and \((1, 2)\) is never a saddle point. So the answer is \(-2 \leq z \leq 1\).

**Problem 4:** (Morris 2.1.5) Compute \(\max_i \min_j m_{i,j}\) and \(\min_j \max_i m_{i,j}\) for the following \(M\).

\[
M = \begin{bmatrix}
0 & 1 & 1 & 2 \\
1 & -1 & 3 & 1 \\
2 & 0 & 0 & 2 \\
3 & 2 & 1 & -1
\end{bmatrix}
\]
**Solution:** Here $\max_i \min_j m_{i,j}$ is achieved by (1,1), (3,2), and (3,3), which have the value 0. Here $\min_j \max_i m_{i,j}$ is achieved by (4,2), (1,4), and (3,4), which have the value 2.

**Problem 5:** (Morris 2.2.5) Consider the mixed strategy game for the matrix

$$M = \begin{bmatrix}
-1 & 2 & -2 & 0 & 1 \\
-2 & -1 & 3 & 2 & 0 \\
2 & 1 & 0 & -1 & -2 \\
0 & 0 & 2 & 1 & 1 \\
1 & -1 & 0 & -2 & 1
\end{bmatrix}$$

a) Use MATLAB’s linear programming solver to solve $\max_\vec{p} \min_\vec{q} E(\vec{p}, \vec{q})$ with the LP formulation from lecture. Write the optimal $\vec{p}^*$ and the optimal objective function value.

a) Use MATLAB’s linear programming solver to solve $\min_\vec{q} \max_\vec{p} E(\vec{p}, \vec{q})$ with the LP formulation from lecture. Write the optimal $\vec{q}^*$ and the optimal objective function value.

**Solution:** See the accompanying code to execute the MATLAB commands.

$$\vec{p}^* = \begin{bmatrix}
\frac{5}{52} \\
0 \\
\frac{11}{52} \\
\frac{17}{26} \\
\frac{1}{26}
\end{bmatrix} \quad \vec{q}^* = \begin{bmatrix}
\frac{21}{52} \\
\frac{2}{13} \\
0 \\
\frac{3}{52} \\
\frac{4}{13}
\end{bmatrix} \quad \text{optimal objective} = \frac{19}{52}. $$