Problem 1: Consider the following nonlinear programming problem (P):
\[
\begin{align*}
\min & \quad \frac{x_1^2}{2} + \frac{x_2^2}{2} \\
\text{s.t.} & \quad x_1 + x_2 + 1 \leq 0.
\end{align*}
\]
Part a and Part b must be done without first consulting your solutions to Homework 4.

a) Write the Lagrangian dual (DP) for this problem and simplify it until it is no longer a nested optimization problem (i.e. not a sup inf problem, but simply a straightforward maximization problem in the (single) dual variable \( \lambda \)).

b) Solve this straightforward maximization problem, finding the optimal \( \lambda \) and the associated optimal objective function value for (DP).

c) Directly using the work from Homework 4, compute the optimal objective function value of (P). Compare to the optimal objective function value of (DP).

d) Provide a saddle point for the Lagrangian function associated with (P) and argue that indeed it is a saddle point.

Problem 2: Suppose \( A \in \mathbb{R}^{n \times n} \) is symmetric positive definite, \( b \in \mathbb{R}^n \) is nonzero, \( c \in \mathbb{R} \) is positive, and the variables \( x \in \mathbb{R}^n \). Consider the following nonlinear programming problem (P):
\[
\begin{align*}
\min & \quad \frac{1}{2} x^T A x \\
\text{s.t.} & \quad b^T x + c \leq 0.
\end{align*}
\]
Part a and Part b must be done without first consulting your solutions to Homework 4.

a) Write the Lagrangian dual (DP) for this problem and simplify it until it is no longer a nested optimization problem (i.e. not a sup inf problem, but simply a straightforward maximization problem in the (single) dual variable \( \lambda \)).

b) Solve this straightforward maximization problem, finding the optimal \( \lambda \) and the associated optimal objective function value for (DP).

c) Directly using the work from Homework 4, compute the optimal objective function value of (P). Compare to the optimal objective function value of (DP).

d) Provide a saddle point for the Lagrangian function associated with (P) and argue that indeed it is a saddle point.
Problem 3: Suppose $S$ is a convex set and, for each of $i = 1, 2, 3, \ldots, k$, the function $f_i(x) : S \to \mathbb{R}$ is convex. Show that the function $a_1f_1(x)+a_2f_2(x)+\cdots+a_kf_k(x)$ is convex, where $a_1, a_2, a_3, \ldots, a_k$ are nonnegative real numbers.

Problem 4:

a) Give a specific example of a particular program of the form $(P) \min f(x) \text{ s.t. } \vec{g}(x) \leq \vec{0}, x \in S$ such that the associated Lagrangian has more than one saddle point.

b) Prove that if a program $(P) \min f(x) \text{ s.t. } \vec{g}(x) \leq \vec{0}, x \in S$ has two saddle points, $x' \in S$, $\lambda' \geq \vec{0}$ and also $x'' \in S$, $\lambda'' \geq \vec{0}$, then $L(x', \lambda') = L(x'', \lambda'')$. 