Homework 3, 553.362 Optimization, Spring 2019

You may discuss ideas for the homework with other students in the class, but when it comes time to write up your solutions you must do it alone, and your write-up must exactly reflect your understanding. You may not consult other textbooks, online resources, nor any existing solutions.

**Problem 1:** Consider a league in the Minor League Baseball system with 14 teams and 42 slots for play over the season. Say that for each slot $i = 1, 2, \ldots, 42$, team $j = 1, 2, \ldots, 14$, and stadium $k = 1, 2, \ldots, 14$, the variable $x_{ijk}$ is 1 or 0 according as team $j$ is in stadium $k$ during slot $i$. Write equations to enforce that no team may play against any other team in back-to-back slots. (So, for example, if team 11 visits team 12 in the 9th slot then team 11 can’t visit team 12 in the 10th slot, nor can team 12 visit team 11 in the 10th slot). This must be accomplished by creating exactly 3731 equations. Also, write equations to enforce that any team cannot revisit any other team until at least 3 slots have passed in between visits. (So, for example, if team 11 visits team 12 in the 30th slot then team 11 can’t revisit team 12 until the 34th slot).

**Solution:** For each $i = 1, 2, \ldots, 41$, then $j = 1, 2, \ldots, 14$, then $k = j+1, j+2, \ldots, 14$ we make an equation $x_{i,j,k} + x_{i,k,j} + x_{i+1,j,k} + x_{i+1,k,j} \leq 1$. Next, for each $i = 1, 2, \ldots, 39$, then $j = 1, 2, \ldots, 14$, then $k = 1, 2, \ldots, 14$ such that $k \neq j$, we make an equation $x_{i,j,k} + x_{i+1,j,k} + x_{i+2,j,k} + x_{i+3,j,k} \leq 1$.

**Problem 2:** Recall that when we were discussing controlling distance travelled in baseball league scheduling, we needed to work with a product $x \cdot y$, where $x$ and $y$ were each binary variables. Unfortunately, this introduced a nonlinearity. What we proposed to do was to created an artificial binary variable $z$ to take the place of $x \cdot y$, with the additional constraint that $x + y - z \leq 1$. Also recall that we were concerned (due to some specific circumstances) that $z$ might take the value 1 even when not both $x$ and $y$ are 1. Your task now is to add linear constraint(s) which will force $z$ to be 1 if and only if both $x$ and $y$ are 1.

**Solution:** Besides $x + y - z \leq 1$, also include $z - x \leq 0$ and $z - y \leq 0$.

**Problem 3:** Suppose that $G$ and $H$ are graphs on $n$ vertices, with respective adjacency matrices $A$ and $B$. Express the graph matching problem as $\min c^T x \text{ s.t. } Mx = b$, $x \in \{0, 1\}^{3n^2}$. Specifically, write down $M$, $b$, and $c$ in terms of $A$, $B$, identity matrices, matrices of 0s, vectors of 1s, vectors of 0s, and Kroenecker products $\otimes$. 

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Solution:

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M = \begin{bmatrix}
I_{n \times n} \otimes A - B^T \otimes I_{n \times n} & -I_{n^2 \times n^2} & I_{n^2 \times n^2} \\
I^T \otimes I_{n \times n} & 0_{n \times n^2} & 0_{n \times n^2} \\
I_{n \times n} \otimes I^T & 0_{n \times n^2} & 0_{n \times n^2}
\end{bmatrix}
\]

\[
b = \begin{bmatrix}
\vec{0} \times n^2 \\
\vec{I} \times n \\
\vec{I} \times n^2
\end{bmatrix}
\]

\[
c = \begin{bmatrix}
\vec{0} \times n^2 \\
\vec{I} \times n^2
\end{bmatrix}
\]

Problem 4: Consider the following experiment: Generate a random 9-by-9 matrix of random integers from \{1, 2, 3, \ldots, 9\}; for example, in MATLAB use command \texttt{M=ceil(9*rand(9,9))}. Next, use your code from last week to find the Sudoku table \(MM\) that agrees with \(M\) in as many places as possible. Record the number of positions where \(MM\) does not agree with \(M\).

Repeat the above experiment 10,000 times, and report the mean and standard deviation of the number of disagreements between \(M\) and \(MM\), over these 10,000 experiments.

Solution: I obtained a mean 40.27 disagreements with standard deviation 1.69 disagreements.