Problem 1: (10 points) In a particular semester, a university offers classes 1, 2, ..., n which are taught at different times. For each pair of classes $i, i'$, denote by $d_{i,i'}$ the number of students who are enrolled in both classes. There are $n$ sequential final exam slots 1, 2, ..., $n$.

Formulate as an integer program the problem of bijectively assigning the classes to final exam slots so as to minimize the number of instances where a student has two consecutive exams. (That is, define the appropriate variables, and write out the constraints and objective function; use summation notation and any appropriate quantifiers.)

Solution: For each $i = 1, 2, ..., n$ and $j = 1, 2, ..., n$, there is a binary variable $x_{i,j}$; value 1 would signal that class $i$ is assigned to the $j$th final exam slot. Also, for each $i = 1, 2, ..., n$ and each $i' = 1, 2, ..., n$ such that $i \neq i'$, and each $j = 1, 2, ..., n-1$, there is a binary variable $y_{i,i',j}$; value 1 would signal that class $i$ is assigned to the $j$th final exam slot and class $i'$ is assigned to the $j+1$th final exam slot. With these variables, the problem is:

$$\text{minimize } \sum_{i=1}^{n} \sum_{j=1,j \neq i}^{n-1} d_{i,i'} y_{i,i',j}$$

s.t. for all $i = 1, 2, ..., n$ it holds that $\sum_{j=1}^{n} x_{i,j} = 1$,

and for all $j = 1, 2, ..., n$ it holds that $\sum_{i=1}^{n} x_{i,j} = 1$,

and for all $i = 1, 2, ..., n$, all $i' = 1, 2, ..., n$ such that $i' \neq i$,

all $j = 1, 2, ..., n-1$ it holds that $x_{i,j} + x_{i',j+1} - y_{i,i',j} \leq 1$.

Problem 2: (10 points) Consider the program (P) minimize $f(x)$ such that $\vec{g}(x) \leq \vec{0}$ and $x \in S$, where $S \subseteq \mathbb{R}^n$ and $f, g_1, g_2, \ldots, g_m : S \rightarrow \mathbb{R}$. Consider any $\vec{x} \in S$ and $\vec{\lambda} \geq \vec{0}_m$. Write down three equivalent ways to say that $\vec{x}, \vec{\lambda}$ is a saddle point of the Lagrangian—no proofs necessary here.

Solution:

$$L(\vec{x}, \lambda) \leq L(\vec{x}, \vec{\lambda}) \leq L(x, \vec{\lambda})$$ for all $x \in S$, $\lambda \geq \vec{0}_m$

$$L(\vec{x}, \vec{\lambda}) \leq L(x, \vec{\lambda})$$ for all $x \in S$, and $\vec{g}(\vec{x}) \leq \vec{0}_m$, and $\vec{\lambda}^T \vec{g}(\vec{x}) = 0$

$\vec{x}$ is optimal in (P), $\vec{\lambda}$ is optimal in the dual of P, and there is no duality gap.
**Problem 3:** (10 points) Consider a simple graph $G = (V, E)$, where $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_m\}$, and suppose that $A \in \{0, 1\}^{m \times n}$ is the incidence matrix; that is, for all $i, j$, $A_{ij}$ is 1 or 0 according as $e_i$ has endpoint $v_j$ or not.

a) Suppose $x \in \{0, 1\}^n$ is an indicator vector for a subset $S \subseteq V$. For any $i$, give a combinatorial interpretation for $(Ax)_i$. (Short answer.)

b) Suppose $y \in \{0, 1\}^m$ is an indicator vector for a subset $F \subseteq E$. For any $i$, give a combinatorial interpretation for $(A^Ty)_i$. (Short answer.)

c) Give a combinatorial interpretation for the integer program (IP) minimize $\mathbf{1}^T x$ s.t. $Ax \geq \mathbf{1}_m$, $x \in \{0, 1\}^n$. (Short answer.)

d) Give a combinatorial interpretation for the dual integer program (DP) maximize $\mathbf{1}^T y$ s.t. $A^Ty \leq \mathbf{1}_n$, $y \in \{0, 1\}^m$. (Short answer.)

e) Order IP, DP and their relaxations by optimal objective function value (short answer).

f) Give a class of graphs where all of the programs IP, DP, and their relaxations have the same objective function value, and explain how you know this.

**Solution:**

a) The number of vertices of $S$ that are endpoints of edge $e_i$.

b) The number of edges of $F$ that saturate the vertex $v_i$.

c) Minimum cardinality vertex cover for $G$; i.e. $\beta(G)$.

d) Maximum cardinality matching for $G$; i.e. $\alpha'(G)$.

e) Let (LP) denote the relaxation of (IP), let (DLP) denote the relaxation of (DP). The optimal objective functions are ordered $\text{oofv(DP)} \leq \text{oofv(DLP)} = \text{oofv(LP)} \leq \text{oofv(IP)}$.

f) For bipartite graphs we have the Konig-Egervary Theorem which asserts that $\alpha'(G) = \beta(G)$, so $\text{oofv(DP)} = \text{oofv(IP)}$, hence the inequalities from part e all become equalities.

**Problem 4:** (10 points) Consider the nonlinear program (P) min $f(x)$ s.t. $\bar{g}(x) \leq \bar{0}$, $x \in S$. Prove that (P) is equivalent to $\inf_{x \in S} \sup_{\lambda \geq 0} L(x, \lambda)$.

**Solution:** $\inf_{x \in S} \sup_{\lambda \geq 0} L(x, \lambda)$ can be expressed $\inf_{x \in S} \phi(x)$ where $\phi(x) := \sup_{\lambda \geq 0} f(x) + \lambda^T \bar{g}(x)$. If $x$ satisfies $\bar{g}(x) \leq \bar{0}$ then for any $\lambda \geq \bar{0}$ it holds that $\lambda^T \bar{g}(x) \leq 0$, so an optimal $\lambda \geq \bar{0}$ for the supremum is $\bar{0}$, hence $\phi(x) = f(x)$. On the other hand, if $\bar{g}(x) \not\leq \bar{0}$, say $g_i(x) > 0$, then letting $\lambda_i \to \infty$ (while the rest of $\lambda$ is 0) yields $\phi(x) = \infty$. Thus

$$\inf_{x \in S} \sup_{\lambda \geq 0} L(x, \lambda) \equiv \inf_{x \in S} \left\{ \begin{array}{ll} f(x) & \text{if } \bar{g}(x) \leq \bar{0} \\ \infty & \text{if } \bar{g}(x) \not\leq \bar{0} \end{array} \right\} \equiv \min f(x) \text{ s.t. } \bar{g}(x) \leq \bar{0}, x \in S$$
Problem 5: (10 points) Consider the linear program \( \min c^T x \) such that \( Ax \geq b, \ x \geq 0 \), where \( A \in \mathbb{R}^{m \times n}, \ b \in \mathbb{R}^m, \ c \in \mathbb{R}^n \), the variables \( x \in \mathbb{R}^n \), and any of the constraints might be active. Write down and simplify what it means for \( \tilde{x}, \tilde{\lambda} \) to be a KKT point and derive from it a condition for optimality that we encountered when studying linear programming. (Hint: You should first express the problem in the form we use for nonlinear programs.)

(See lecture notes at end of exam material)

Problem 6: (10 points) Suppose \( A \in \mathbb{R}^{n \times n} \) is symmetric positive definite, \( b \in \mathbb{R}^n \) is nonzero, \( c \in \mathbb{R} \) is positive, and the variables \( x \in \mathbb{R}^n \). Consider (P): \( \min \frac{1}{2} x^T Ax \) s.t. \( b^T x + c \leq 0 \). Write the Lagrangian dual (DP) \( \sup_{\lambda \geq 0} \theta(\lambda) \); your task is to simplify \( \theta \) as much as possible.

Solution: The Lagrangian function is \( L(x, \lambda) = \frac{1}{2} x^T Ax + \lambda (b^T x + c) \) for all \( x \in \mathbb{R}^n, \ \lambda \geq 0 \).

(DP) is \( \sup_{\lambda \geq 0} \theta(\lambda) \) where, for any fixed \( \lambda \geq 0 \), we have that \( \theta(\lambda) := \inf_{x \in \mathbb{R}^n} \frac{1}{2} x^T Ax + \lambda (b^T x + c) \).

This unconstrained minimization is easy to solve, since the objective function is convex (since the Hessian is always \( A \), which is positive definite), so we set its gradient \( Ax + \lambda b \) to the zero vector and solve for \( x \), yielding for this minimization problem the global minimum \( x = -\lambda A^{-1} b \) (note that \( A \) is positive definite and thus invertible), which has objective function value \( \frac{\lambda^2}{2} b^T A^{-1} b - \lambda^2 b^T A^{-1} b + \lambda c \).

We simplify this to obtain \( \theta(\lambda) = -\frac{\lambda^2}{2} b^T A^{-1} b + \lambda c \). Thus (DP) may be expressed as the simple maximization problem \( \sup_{\lambda \geq 0} -\frac{\lambda^2}{2} b^T A^{-1} b + \lambda c \).
This page will not be graded.