Problem 1: Geometrically solve the following LP:

$$\begin{align*}
\text{min} & \quad x_1 - 2x_2 \\
\text{s.t.} & \quad x_1 - x_2 \geq -4 \\
& \quad -3x_1 - 2x_2 \geq -18 \\
& \quad -3x_1 + x_2 \geq -9 \\
& \quad x_1, x_2 \geq 0
\end{align*}$$

Again geometrically solve for each of the LPs with the same feasible region as above but with respective objective functions $-x_1 + 2x_2$, and $-3x_1 - x_2$, and $3x_1 + x_2$, and $3x_1 - 3x_2$. (Five different LPs should be solved, but you can render a single plotting of the feasible region.)

Solution: The feasible region is bounded by the lines (clockwise from the origin) $x_1 = 0$, $x_2 = x_1 + 4$, $x_2 = -\frac{3}{2}x_1 + 9$, $x_2 = 3x_1 - 9$, and $x_2 = 0$ which have respective slopes of $\infty$, $1$, $-\frac{3}{2}$, $3$, $0$, and the vertices of the polytope are $(0, 0)$, $(0, 4)$, $(2, 6)$, $(4, 3)$, $(3, 0)$, clockwise from the origin. The level sets for the respective objective functions have slopes $\frac{1}{2}$ with lower objective at higher $x_2$ intercepts, slopes $\frac{1}{2}$ with lower objective at lower $x_2$ intercepts, slopes $-3$ with lower objective at higher $x_2$ intercepts, slopes $-3$ with lower objective at lower $x_2$ intercepts, and slopes $1$ with lower objective at higher $x_2$ intercepts. Thus the solutions are: first problem $(2, 6)$, second problem $(3, 0)$, third problem $(4, 3)$, fourth problem $(0, 0)$, and fifth problem are all points in line segment joining $(0, 4)$ to $(2, 6)$.

Problem 2: Recall the nonlinear optimization problem which we have previously considered:

$$\begin{align*}
\text{max} & \quad 1 + x_1^2(x_2 - 1)^3e^{-x_1-x_2} \\
\text{s.t.} & \quad x_2 \geq \log x_1 \\
& \quad x_1 + x_2 \leq 6 \\
& \quad x_1, x_2 \geq 0
\end{align*}$$

We are interested in finding out where $x_1 + x_2 = 6$ intersects $x_2 = \log x_1$ (i.e. for what values $x_1, x_2$ these equations are simultaneously true). Note that $x_1$ would solve $x_1 = 6 - \log x_1$; this is called fixed point form since for the function $f(x) = 6 - \log x$ it would hold that $x_1 = f(x_1)$. Do the following in MATLAB: Start with any value $z$ which you guess is close to $x_1$, then evaluate...
\( f(z), f(f(z)), f(f(f(z))), \) [each time feeding the output of \( f \) back into \( f \) in a loop] until the sequence seems to converge... and this sequence converges to \( x_1 \) if all goes well. (This can be done interactively in MATLAB; hand in the diary with the important parts of it circled.)

**Solution:** The following commands in MATLAB will do the work we need:

```matlab
>> 6-log(3)
ans = 4.9014

>> 6-log(ans)
ans = 4.4105

>> 6-log(ans)
ans = 4.5160

>> 6-log(ans)
ans = 4.4924

>> 6-log(ans)
ans = 4.4976

>> 6-log(ans)
ans = 4.4965

>> 6-log(ans)
ans = 4.4967

>> 6-log(ans)
ans = 4.4967

>> format long

>> 6-log(ans)
ans = 4.49666365333577

>> x1=ans;

>> x2=log(x)
ans = 1.50333571142585

>> log(x)+x
ans = 5.99999936476163
```

so the point of intersection is 4.49666,1.50333.
Problem 3: Recall the nonlinear optimization problem which we have previously considered:

$$\text{max} \quad 1 + x_1^2(x_2 - 1)^3 e^{-x_1-x_2}$$

s.t. \quad x_2 \geq \log x_1 \\
\quad x_1 + x_2 \leq 6 \\
\quad x_1, x_2 \geq 0

Suppose that we further restricted the feasible region by additionally requiring that $x_1 + x_2 = 6$. Use calculus to solve the problem exactly. (Note that you can turn this into a problem in a single variable. You can use MATLAB to find the roots of a polynomial if this is helpful.)

Solution: Effectively, we are maximizing $f(x) := x^2(5 - x)^3 = -x^5 + 15x^4 - 75x^3 + 125x^2$ for $x := x_1$, and we compute the derivative $f'(x) = -5x^4 + 60x^3 - 225x^2 + 250x$ which has roots at 0, 2, and 5 (5 is a repeated root). Note that $f'(1) = 80$, $f'(3) = -60$, $f'(6) = -120$ thus, on the interval $[0, 6]$, the function increases till $x = 2$, then decreases. Thus global maximum is at $x = 2$; so $x_1 = 2, x_2 = 4$ is global max of the problem of interest.

Problem 4: (This problem comes from Winston and Venkataramanan, originally from Sullivan and Secrest.) Lizzie’s Dairy produces cream cheese and cottage cheese. Milk and cream are blended to produce these two products. Both high-fat and low-fat milk can be used to produce cream cheese and cottage cheese. High-fat milk is 60% fat; low-fat milk is 30% fat. The milk used to produce cream cheese must average at least 50% fat, and that for cottage cheese at least 35% fat. At least 40% (by weight) of the inputs to cream cheese and at least 20% (by weight) of the inputs to cottage cheese must be cream. Both cream cheese and cottage cheese are produced by putting milk and cream through the cheese machine. It costs $0.40 to process 1 lb of inputs into into a pound of cream cheese. It costs $0.40 to produce 1 lb of cottage cheese, but every pound of input for cottage cheese yields 0.9 lb of cottage cheese and 0.1 lb of waste. Cream is produced by evaporating high-fat and low-fat milk. It costs $0.40 to evaporate 1 lb of high-fat milk, and each pound of high-fat milk that is evaporated yields 0.6 lb of cream. It costs $0.40 to evaporate 1 lb of low-fat milk, and each pound of low-fat milk that is evaporated yields 0.3 lb of cream. Each day, up to 3000 lb of input may be sent through the cheese machine. Each day, at least 1000 lb of cream cheese and 1000 lb of cottage cheese must be produced. Up to 1500 lb of cream cheese and 2000 lb of cottage cheese can be sold each day. Cream cheese is sold for $1.50 per lb and cottage cheese for $1.20 per lb. High-fat milk is purchased for $0.80 per lb, and low-fat milk for $0.40 per lb. The evaporator can process at most 2000 lb of milk daily. Formulate a linear
program in canonical form that can be used to maximize Lizzie’s daily profit.
(In working on this problem, provide the matrix $A$ and vectors $b$ and $c$ for the LP. Declare

- $x_1$ = lb of high-fat milk that will go to cream cheese in cheese machine
- $x_2$ = lb of low-fat milk that will go to cream cheese in cheese machine
- $x_3$ = lb of cream that will go to cream cheese in cheese machine
- $x_4$ = lb of high-fat milk that will go to cottage cheese in cheese machine
- $x_5$ = lb of low-fat milk that will go to cottage cheese in cheese machine
- $x_6$ = lb of cream that will go to cottage cheese in cheese machine
- $x_7$ = lb of high-fat milk that will go to cream
- $x_8$ = lb of low-fat milk that will go to cream.)

Solution: First, in rough form. The objective function is

$$0.4(x_1 + x_2 + x_3) + 0.4(0.9)(x_4 + x_5 + x_6) +$$

$$0.4x_7 + 0.4x_8 - 1.5(x_1 + x_2 + x_3) - 1.2(0.9)(x_4 + x_5 + x_6) +$$

$$0.8(x_1 + x_4 + x_7) + 0.4(x_2 + x_5 + x_8).$$

The constraints are nonnegativity of variables and

- $0.6x_1 + 0.3x_2 \geq 0.5(x_1 + x_2)$
- $0.6x_4 + 0.3x_5 \geq 0.35(x_4 + x_5)$
- $x_3 \geq 0.4(x_1 + x_2 + x_3)$
- $x_6 \geq 0.2(x_4 + x_5 + x_6)$
- $0.6x_7 + 0.3x_8 \geq x_3 + x_6$
- $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 3000$
- $1000 \leq x_1 + x_2 + x_3 \leq 1500$
- $1000 \leq 0.9(x_4 + x_5 + x_6) \leq 2000$
- $x_7 + x_8 \leq 2000$
In matrix form,

\[
A = \begin{bmatrix}
0.1 & -0.2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.25 & -0.05 & 0 & 0 & 0 \\
-0.4 & -0.4 & 0.6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.2 & -0.2 & 0.8 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & -1 & 0.6 & 0.3 \\
-1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.9 & 0.9 & 0.9 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.9 & 0.9 & 0.9 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\
\end{bmatrix},
\]

\[
b = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
-3000 \\
1000 \\
-1500 \\
1000 \\
-2000 \\
-2000 \\
\end{bmatrix},
\]

\[
c = \begin{bmatrix}
-0.3 \\
-0.7 \\
-1.1 \\
0.08 \\
-0.32 \\
-0.72 \\
1.2 \\
0.8 \\
\end{bmatrix}.
\]