Asymptotic Analysis via Mellin Transforms for Small Deviations in L^2-norm of Integrated Brownian Sheets

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ABSTRACT

We use Mellin transforms to compute a full asymptotic expansion for the tail of the Laplace transform of the squared L^2-norm of any multiply-integrated Brownian sheet. Through reversion we obtain corresponding strong small-deviation estimates.

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1 Introduction

Throughout, for readability, we often use shorthand vector notation such as $\mathbf{t}$ for $(t_1, \ldots, t_d)$ and $d\mathbf{t}$ for $dt_1 \cdots dt_d$.

A d-parameter (standard) *Brownian sheet* $B = \big(B(\mathbf{t}) : \mathbf{t} \in [0,1]^d\big)$ is defined to be a real-valued Gaussian random field with continuous sample paths, mean function $\mathbf{E}\, B(\mathbf{t}) = 0$, and covariance kernel $K_{\mathbf{0}}(\mathbf{s}, \mathbf{t}) := \mathbf{E}[B(\mathbf{s})B(\mathbf{t})] = \prod_{j=1}^{d} \min\{s_j, t_j\}$ for $\mathbf{s}, \mathbf{t} \in [0,1]^d$. Notice that the covariance operator with kernel $K_{\mathbf{0}}$ is the tensor product of d copies of the covariance operator of Brownian motion. Let $\mathbf{m} = (m_1, \ldots, m_d)$ be a fixed d-vector with nonnegative integer components. We define $\mathbf{m}$*-integrated Brownian sheet* $X_{\mathbf{m}}$ by

$$X_{\mathbf{m}}(\mathbf{t}) := \int_0^{t_1} \cdots \int_0^{t_d} \prod_{j=1}^{d} \frac{(t_j - u_j)^{m_j}}{m_j!} \, B(du_1, \ldots, du_d). \tag{1.1}$$

It follows immediately that $X_{\mathbf{m}}$ is a mean-zero Gaussian random field on $[0,1]^d$ with covariance kernel

$$K_{\mathbf{m}}(\mathbf{s}, \mathbf{t}) = \prod_{j=1}^{d} \int_0^{\min\{s_j, t_j\}} \frac{(s_j - u_j)^{m_j}(t_j - u_j)^{m_j}}{(m_j!)^2} \, du_j,$$

and so the covariance operator of $X_{\mathbf{m}}$ is simply the tensor product of the covariance operators with kernels K_{m_j}, which are the covariance kernels of m_j-integrated Brownian motions. Observe that $X_{\mathbf{0}}$ is simply d-parameter Brownian sheet B.

Remark 1.1. To motivate the definition of an integrated Brownian sheet, recall that m-times integrated Brownian motion X_m can be defined (naturally enough) in terms of Brownian motion B by

$$X_m(t) := \int_0^t \int_0^{s_m} \cdots \int_0^{s_2} B(s_1) \, ds_1 \cdots ds_m \tag{1.2}$$

for integer $m \geq 1$, with $X_0 := B$. It is not difficult to see that X_m has the same distribution as the process with value at time t given by

$$\frac{1}{m!} \int_0^t (t - u)^m \, dB(u). \tag{1.3}$$

That is, the m integrations in (1.2) can be collapsed to the one in (1.3). Similar motivation can be given for our definition of $\mathbf{m}$-integrated Brownian sheet.

Consider the squared L^2-norm of $X_{\mathbf{m}}$:

$$V^2 \equiv V^2_{\mathbf{m}} := \int_0^1 \cdots \int_0^1 X^2_{\mathbf{m}}(\mathbf{t}) \, d\mathbf{t}. \tag{1.4}$$

The classical Karhunen–Loève expansion tells us that $X_{\mathbf{m}}$ has the same distribution as the process $\big(\sum_{\mathbf{n}} \sqrt{a_{\mathbf{n}}} \varphi_{\mathbf{n}}(\mathbf{t}) \xi_{\mathbf{n}}\big)$, where the $\xi_{\mathbf{n}}$'s are i.i.d. standard normal random variables and the $\varphi_{\mathbf{n}}$'s form a complete orthonormal system of eigenvectors, with corresponding eigenvalues $a_{\mathbf{n}}$, for the covariance operator $\mathcal{A}_{\mathbf{m}} : L^2([0,1]^d) \to L^2([0,1]^d)$. The spectrum $\sigma(\mathcal{A}_{\mathbf{m}}) = \{a_{\mathbf{n}}\}$ is the product of spectra of the covariance operators $\mathcal{A}_{m_j}$ of the associated m_j-integrated Brownian motions:

$$\sigma(\mathcal{A}_{\mathbf{m}}) = \sigma(\mathcal{A}_{m_1}) \otimes \cdots \otimes \sigma(\mathcal{A}_{m_d}),$$

where $\otimes$ represents elementwise set multiplication: $S \otimes T := \{st : s \in S \text{ and } t \in T\}$. It therefore follows that, in distribution,

$$V^2 = \sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} a_{\mathbf{n}} \xi_{\mathbf{n}}^2, \tag{1.5}$$

where $a_{\mathbf{n}} = a_{\mathbf{n}}(\mathbf{m}) = a_{n_1}(m_1) \cdots a_{n_d}(m_d)$ and $(a_{n_j}(m_j))$ are the eigenvalues for $\mathcal{A}_{m_j}$.

We are interested in deriving a "strong" small-deviations result, i.e., in computing the lead-order asymptotics as $\varepsilon \to 0+$ for the small-ball probability $\mathbf{P}(V^2 \le \varepsilon)$ (not just for its logarithm). In light of the representation (1.5), this can be done using Sytaja's Tauberian Theorem [15]:

Theorem 1.2 (Sytaja [15]). *Suppose that $a_n > 0$ for all n with $\sum_{n=1}^{\infty} a_n < \infty$, and that $(\xi_n)_{n \ge 1}$ are i.i.d. standard normal random variables. Then as $\varepsilon \to 0+$,*

$$\mathbf{P}\left(\sum_{n=1}^{\infty} a_n \xi_n^2 \le \varepsilon\right) = (1 + o(1)) \left[-2\pi (x^*)^2 h''(x^*)\right]^{-1/2} \exp\left\{-[h(x^*) - \varepsilon\, x^*]\right\}$$

where $h(x) := -\log \mathbf{E} \exp\left\{-x \sum_{n=1}^{\infty} a_n \xi_n^2\right\}$, $x \ge 0$, denotes the log Laplace transform and $x^* \equiv x^*(\varepsilon)$ is defined implicitly in terms of ε:

$$h'(x^*) = \varepsilon. \quad \square \tag{1.6}$$

In this paper we first show how to derive, for arbitrary $\mathbf{m}$, a complete asymptotic expansion as $x \to \infty$ for the log Laplace transform of V^2, namely, $h_{\mathbf{m}}(x) := -\log \mathbf{E} \exp\{-x V_{\mathbf{m}}^2\}$, and for each of its derivatives. Our method proceeds in two steps. First we study carefully the one-dimensional functions h_{m_j}. Then we apply Mellin-transform techniques that prove to be quite powerful in handling our "tensored" processes. Although Mellin transforms enjoy common use in the analysis of algorithms [4], their application here is, to our knowledge, the first in studying small or large deviations. As the reader will see, their use is very natural and makes many computations completely transparent.

We also show how one can develop the explicit asymptotic behavior of x^* in terms of ε via the relation (1.6). This "reversion" can be quite nontrivial and is the subject of Section 4. Once this reversion is understood, one can then easily apply Theorem 1.2 to obtain the strong small-ball asymptotics.

If X_m is m-times integrated Brownian motion, then the lead-order asymptotics of $-\log \mathbf{P}(V^2 \le \varepsilon)$ have been studied by Chen and Li [1]. In this one-dimensional case, lead-order asymptotics for the small-ball probability $\mathbf{P}(V^2 \le \varepsilon)$ itself have been studied by Gao et al. ([5], [6], and [7]), Nazarov [13], and Nazarov and Nikitin [14]. A closed-form expression for the Laplace transform of V_m^2 for any m has been derived in Gao et al. [7]; we should also mention that in the special case $m = 1$ the Laplace transform of (1.4) had been obtained explicitly in Khoshnevisan and Shi [10] several years earlier.

We now turn our attention to higher dimensions. For $d \geq 1$, Csaki [3] computes the lead-order asymptotics of $-\log \mathbf{P}(V_{\mathbf{0}}^2 \leq \varepsilon)$, that is, the lead order for logarithmic small deviations of d-dimensional Brownian sheet. But a different approach is needed for nonzero m_j's. To illustrate, consider (m_1, m_2)-integrated Brownian sheet. Then

$$\begin{aligned} h(x) &= -\log \mathbf{E} \exp\left\{ -x \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} a_{n_1,n_2} \xi_{n_1,n_2}^2 \right\} \\ &= \frac{1}{2} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \log\left(1 + \frac{2x}{\kappa_{n_1}(m_1)\kappa_{n_2}(m_2)} \right), \end{aligned} \tag{1.7}$$

where $a_{n_1,n_2} = 1/\left[\kappa_{n_1}(m_1)\kappa_{n_2}(m_2)\right]$ and the $\kappa_n(m)$, $n \geq 1$, are the reciprocals of the eigenvalues of $\mathcal{A}_m$. The reason for moving to reciprocals is that the κ's have nice representations as the zeros of certain entire functions [7]. If $m_1 = m_2 = 0$, then $\kappa_n(0) = \left[\left(n - \frac{1}{2}\right)\pi\right]^2$ and Csaki's method works well because there are known expressions and bounds for the summations involved. However, if a component of $\mathbf{m}$ differs from 0 then his approach is untenable. For example, if $m_1 = 0$ and $m_2 = 1$, then $(\kappa_n(1))$ are the solutions to the equation $\cos(z^{1/4})\cosh(z^{1/4}) + 1 = 0$. When n is large, $\kappa_n(1)$ is quite close to $\left[\left(n - \frac{1}{2}\right)\pi\right]^4$ (see, for instance, [5]), but it does not seem possible to obtain explicit expressions. In this case we immediately see the complexity introduced when $m_2 = 1$.

Csaki handled only the case $\mathbf{m} = \mathbf{0}$. Recently, Karol' et al. [9] extended the classical result of Csaki [3] to obtain lead-order logarithmic small deviations for many Gaussian random fields; in particular, their treatment can handle our problem for arbitrary $\mathbf{m}$. However, their methods do not seem to extend to obtaining asymptotic results beyond the lead order for the log small-ball probability.

We note that the methods developed in this paper can be applied to many other Gaussian random fields besides $\mathbf{m}$-integrated Brownian sheet $X_{\mathbf{m}}$. Consider a Gaussian random field whose covariance operator is the tensor product of marginal operators. If the Gaussian processes corresponding to these marginal operators each have a log Laplace transform whose asymptotic behavior can be established, then the methods of this paper will apply. Further, since (at least for $X_{\mathbf{m}}$) we can derive a complete asymptotic expansion for the log Laplace transform, we need only produce a complete asymptotic expansion extending Sytaja's theorem in order to obtain a complete asymptotic expansion—not just the lead-order asymptotics—for the small-ball probability. We will produce the needed generalization of Sytaja's theorem, and apply it to small-ball probabilities for various Gaussian random fields, in future work.

This paper is organized as follows. In Section 2 we provide basic background concerning the Mellin transform. We derive asymptotic expansions for $h(x)$ and its derivatives in Section 3. In Section 4 we discuss explicit expansions for the solution $x^* \equiv x^*(\varepsilon)$ to the equation $h'(x) = \varepsilon$ [recall (1.6)]. Finally, in Section 5 we derive lead-order asymptotics for the small-ball probability.

Without loss of generality we will assume throughout this paper that

$$0 \le m_1 \le m_2 \le \cdots \le m_d < \infty.$$

For multibranched functions of a complex variable, we use the principal branch unless otherwise specified.

2 Mellin transforms

In this section we collect, for the reader's convenience, some useful facts concerning Mellin transforms. Our treatment follows closely the superb book manuscript by Flajolet and Sedgewick [4]. Throughout, we use the following notation for an open vertical strip in the complex plane: for real c, d,

$$\langle c, d\rangle := \{s \in \mathbb{C} : c < \operatorname{Re} s < d\}.$$

2.1 Definition

Suppose that $f : [0,\infty) \to \mathbb{C}$ is locally integrable [i.e., is Lebesgue integrable over any bounded closed subinterval of $(0,\infty)$; this condition is met, for example, if f is continuous]. For those $s \in \mathbb{C}$ such that $x \mapsto f(x)x^{s-1}$ is Lebesgue integrable over $(0,\infty)$, we define the *Mellin transform* f^* of f as the Lebesgue integral

$$f^*(s) := \int_0^\infty f(x)x^{s-1}\,dx. \tag{2.1}$$

Keep in mind that Lebesgue integrability is a form of absolute integrability.

2.2 Existence: the fundamental strip; transform of derivative

Given local integrability, what is at issue for the existence of (2.1) is the behavior of f near 0 and near ∞. It is easy to check that there exists a (possibly empty) maximal open vertical strip in which the integral (2.1) is well defined; that strip is called the *fundamental strip*.

Suppose, for example, that f has the properties

$$f(x) = O(x^b) \quad \text{as } x \to 0+ \qquad \text{and} \qquad f(x) = O(x^a) \quad \text{as } x \to \infty \tag{2.2}$$

with $a < b$. Let α denote the infimum of such a's, and β the supremum of such b's. Then $\langle -\beta, -\alpha\rangle$ is a substrip of the fundamental strip [and equals the fundamental strip in typical cases, such as when $f(x) = (1+o(1))c_0x^\beta$ as $x \to 0+$ and $f(x) = (1+o(1))c_\infty x^\alpha$ as $x \to \infty$ with $c_0, c_\infty \ne 0$]. If, further, f is continuously differentiable and monotone, then one can check using integration by parts that $\langle 1-\beta, 1-\alpha\rangle$ is a substrip of the fundamental strip for f' and that

$$(f')^*(s) = -(s-1)f^*(s-1), \quad s \in \langle 1-\beta, 1-\alpha\rangle. \tag{2.3}$$

In this paper, we will make key use of the following Mellin transform pair:

$$f(x) = \log(1+x) \longleftrightarrow f^*(s) = \frac{\pi}{s\sin(\pi s)}, \tag{2.4}$$

with fundamental strip $\langle -1, 0\rangle$. For justification of the double arrow, see the start of the next subsection.

2.3 Inversion and the mapping property

Suppose for simplicity that f is continuous and has a nonempty fundamental strip, which we denote by $\langle -\beta, -\alpha\rangle$. Then for any c in the interval $(-\beta, -\alpha)$ we have

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f^*(s)x^{-s}\, ds. \tag{2.5}$$

This establishes a correspondence between functions and their Mellin transforms.

Mellin transforms are useful in asymptotic analysis because, under a certain growth condition on the meromorphic continuation of the Mellin transform f^*, there is a correspondence between the asymptotic expansion of the function f near 0 or ∞ and the singularities of $f^*(s)$ for $s \in \mathbb{C}$. (For our purposes, we need only asymptotic expansions near ∞, so those are all we will discuss here.) Flajolet and Sedgewick [4] call this correspondence the *mapping property* of Mellin transforms. To explain more fully, it is convenient to introduce the following formal-sum shorthand notation.

Let $\varphi : \Omega \to \mathbb{C}$ be a meromorphic function in a domain Ω and let $\mathcal{P} \subset \Omega$ be the set of its poles. For each $s_0 \in \mathcal{P}$, let $\Delta_{s_0}(s)$ be a truncation of the Laurent series for $\varphi(s)$ at $s_0 \in \mathcal{P}$; we assume that $\Delta_{s_0}(s)$ contains at least all terms of order $(s - s_0)^{-k}$ with $k > 0$. We then write

$$\varphi(s) \asymp \sum_{s_0\in\mathcal{P}} [\Delta_{s_0}(s)]_{s=s_0} \quad (s \in \Omega), \tag{2.6}$$

or, when no confusion results, simply

$$\varphi(s) \asymp \sum_{s_0\in\mathcal{P}} \Delta_{s_0}(s) \quad (s \in \Omega),$$

and call (2.6) a *singular expansion* of φ in Ω.

The following two theorems are essential for our results. The first of these theorems gives (among other things) sufficient conditions for meromorphic continuation of the Mellin transform f^*. The second theorem shows that a singular expansion of f^* characterizes the asymptotic behavior of f.

Theorem 2.1 (Direct Mapping Theorem). *Let f have Mellin transform f^* at least in a nonempty strip $\langle -\beta, -\alpha\rangle$. Assume that $f(x)$ admits as $x \to \infty$ a finite asymptotic expansion of the form*

$$f(x) = \sum_{(\xi,k)\in A} c_{\xi,k}\, x^{\xi}(\log x)^k + O(x^{\gamma}), \tag{2.7}$$

for a finite set A of pairs (ξ, k), where the ξ's satisfy $\gamma < \xi \le \alpha$ and the k's are nonnegative integers. Then f^* is continuable to a meromorphic function in the strip $\langle -\beta, -\gamma\rangle$, where it admits the singular expansion

$$f^*(s) \asymp -\sum_{(\xi,k)\in A} c_{\xi,k} \frac{(-1)^k k!}{(s+\xi)^{k+1}} \quad (s \in \langle -\beta, -\gamma\rangle). \qquad \Box$$

Theorem 2.2 (Reverse Mapping Theorem). *Let f be continuous in $(0,\infty)$ with Mellin transform f^* existing at least in a nonempty strip $\langle -\beta, -\alpha\rangle$. Assume that $f^*(s)$ admits a meromorphic continuation to the strip $\langle -\beta, -\gamma\rangle$ for some $\gamma < \alpha$ with a finite number of poles there, and is analytic on $\operatorname{Re} s = -\gamma$. Assume also that there exists a real number $\eta \in (\alpha, \beta)$ such that*

$$f^*(s) = O(|s|^{-r}), \quad \text{with} \quad r > 1, \tag{2.8}$$

when $|s| \to \infty$ in $-\eta \le \operatorname{Re} s \le -\gamma$. If f^* admits the singular expansion

$$f^*(s) \asymp -\sum_{(\xi,k)\in A} c_{\xi,k} \frac{(-1)^k k!}{(s+\xi)^{k+1}} \quad (s \in \langle -\eta, -\gamma\rangle)$$

with all k's in the sum nonnegative integers, then an asymptotic expansion of $f(x)$ as $x \to \infty$ is

$$f(x) = \sum_{(\xi,k)\in A} c_{\xi,k}\, x^{\xi} (\log x)^k + O(x^{\gamma}). \qquad \Box$$

2.4 Termwise differentiation of an asymptotic expansion

The proof of the following result is a simple exercise in the use of the two mapping theorems. Note the condition $r > 2$.

Corollary 2.3. *Let f be continuously differentiable and monotone, and let α and β be defined as in the paragraph containing (2.2) and (2.3). Assume that $\alpha < \beta$, that $f(x)$ admits as $x \to \infty$ a finite asymptotic expansion (2.7) as in the statement of the Direct Mapping Theorem, and that f^* has a meromorphic continuation that is analytic on the line $\operatorname{Re} s = -\gamma$ and satisfies the growth condition (2.8) with $r > 2$. Then the expansion (2.7) can be differentiated termwise: as $x \to \infty$,*

$$f'(x) = \sum_{(\xi,k)\in A} c_{\xi,k}\, x^{\xi-1} (\log x)^{k-1} (\xi \log x + k) + O(x^{\gamma-1}). \qquad \Box$$

2.5 Harmonic sums and the separation property

The mapping property discussed in Section 2.3 is particularly effective for asymptotic analysis of a harmonic sum

$$F(x) = \sum_k \lambda_k f(\mu_k x) \tag{2.9}$$

with $\mu_k > 0$ for every k. The reason is the readily checked *separation property* that the Mellin transform of (2.9) has the simple product form

$$F^*(s) = f^*(s) \sum_k \lambda_k \, \mu_k^{-s}, \tag{2.10}$$

valid for s in the intersection of the fundamental strip of f and the domain of absolute convergence of the *generalized Dirichlet series* $\sum_k \lambda_k \mu_k^{-s}$.

3 Asymptotic expansion for the log Laplace transform

Recall the expression (1.7) for the log Laplace transform $h_{\mathbf{m}} \equiv h$ of V^2. To avoid nuisance factors it is convenient for us to reparameterize $h_{\mathbf{m}}$ by defining, for $x \geq 0$,

$$L_{\mathbf{m}}(x) := -2 \log \mathbf{E} \exp\left\{-\frac{x}{2} V^2\right\} = \sum_{n_1=1}^{\infty} \cdots \sum_{n_d=1}^{\infty} \log\left(1 + \frac{x}{\kappa_{n_1}(m_1) \cdots \kappa_{n_d}(m_d)}\right). \tag{3.1}$$

Notice that $h_{\mathbf{m}}(x) = \frac{1}{2} L_{\mathbf{m}}(2x)$. In this section we derive a complete asymptotic expansion for $L_{\mathbf{m}}$, for general $\mathbf{m}$. First we treat the one-dimensional case. Then we bring Mellin asymptotic summation to bear. That is, we factor the Mellin transform $L^*_{\mathbf{m}}$ in terms of the transforms $L^*_{m_j}$ using the separation property, analyze the singularities of the factors using (among other things) the Direct Mapping Theorem, and finally obtain the asymptotic expansion for $L_{\mathbf{m}}$ (or for $h_{\mathbf{m}}$) using the Reverse Mapping Theorem. Using Corollary 2.3 we also derive corresponding expansions for the first two derivatives of $h_{\mathbf{m}}$, as needed for our application in Section 5 of Sytaja's Tauberian Theorem.

3.1 The one-dimensional case

We begin by analyzing m-integrated Brownian motion, i.e., the $d = 1$ case

$$L_m(x) = \sum_{n=1}^{\infty} \log\left(1 + \frac{x}{\kappa_n(m)}\right). \tag{3.2}$$

For integer ℓ, $0 \leq \ell \leq 2m+1$, set $\omega_\ell := \exp\{i\pi \frac{\ell}{m+1}\} = \omega_1^\ell$, $v_\ell := \exp\{i\pi \frac{2\ell+1}{2m+2}\} = v_0 \omega_\ell$, and $\beta_\ell(x) \equiv \beta_\ell := x^{1/(2m+2)} i v_\ell$. We employ the following result from Gao et al. (cf. Theorem 6 in [7]), which gives an exact expression for $L_m(x)$:

Lemma 3.1. *For $x \geq 0$ we have $L_m(x) = \log|\det N(x)| - (m+1)\log(2m+2)$, where*

$$N(x) := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega_0 & \omega_1 & \cdots & \omega_{2m+1} \\ \vdots & \vdots & \cdots & \vdots \\ \omega_0^m & \omega_1^m & \cdots & \omega_{2m+1}^m \\ \omega_0^{m+1} e^{\beta_0} & \omega_1^{m+1} e^{\beta_1} & \cdots & \omega_{2m+1}^{m+1} e^{\beta_{2m+1}} \\ \vdots & \vdots & \cdots & \vdots \\ \omega_0^{2m+1} e^{\beta_0} & \omega_1^{2m+1} e^{\beta_1} & \cdots & \omega_{2m+1}^{2m+1} e^{\beta_{2m+1}} \end{pmatrix}. \qquad \square$$

The following important asymptotic consequence will be needed.

Lemma 3.2. *As $x \to \infty$, the quantity $L_m(x)$ defined at* (3.2) *has the expansion*

$$\begin{aligned} L_m(x) &= \csc\left(\frac{\pi}{2m+2}\right) x^{1/(2m+2)} + 2\log|\det U| - (m+1)\log(2m+2) \\ &\quad + O\left(\exp\left\{-\sin\left(\frac{\pi}{2m+2}\right) x^{1/(2m+2)}\right\}\right), \end{aligned}$$

where $U_m \equiv U$ is the Vandermonde matrix

$$U := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \omega_0 & \omega_1 & \cdots & \omega_m \\ \omega_0^2 & \omega_1^2 & \cdots & \omega_m^2 \\ \vdots & \vdots & \cdots & \vdots \\ \omega_0^m & \omega_1^m & \cdots & \omega_m^m \end{pmatrix}. \tag{3.3}$$

Proof. We need to study the large-x behavior of $N(x)$. Multiply the last $m+1$ columns of $N(x)$ by e^{β_0}, e^{β_1}, ..., e^{β_m}, respectively, and use $\beta_j = -\beta_{m+1+j}$ to obtain the following matrix:

$$\overline{N}(x) := \begin{pmatrix} 1 & \cdots & 1 & e^{\beta_0} & \cdots & e^{\beta_m} \\ \omega_0 & \cdots & \omega_m & \omega_{m+1}e^{\beta_0} & \cdots & \omega_{2m+1}e^{\beta_m} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \omega_0^m & \cdots & \omega_m^m & \omega_{m+1}^m e^{\beta_0} & \cdots & \omega_{2m+1}^m e^{\beta_m} \\ \omega_0^{m+1}e^{\beta_0} & \cdots & \omega_m^{m+1}e^{\beta_m} & \omega_{m+1}^{m+1} & \cdots & \omega_{2m+1}^{m+1} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ \omega_0^{2m+1}e^{\beta_0} & \cdots & \omega_m^{2m+1}e^{\beta_m} & \omega_{m+1}^{2m+1} & \cdots & \omega_{2m+1}^{2m+1} \end{pmatrix}.$$

For $0 \le \ell \le m$ we have

$$|e^{\beta_\ell}| \le |e^{\beta_0}| = \exp\{-\sin(\pi/(2m+2))\, x^{1/(2m+2)}\}.$$

Using the permutation expansion of the determinant, we find

$$\det \overline{N}(x) = \det \overline{N}(\infty) + O(|e^{\beta_0}|),$$

where $\overline{N}(\infty)$ is the matrix obtained from $\overline{N}(x)$ by replacing each entry with a factor e^{β_ℓ} by 0:

$$\overline{N}(\infty) = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 \\ \omega_0 & \cdots & \omega_m & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega_0^m & \cdots & \omega_m^m & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \omega_{m+1}^{m+1} & \cdots & \omega_{2m+1}^{m+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \omega_{m+1}^{2m+1} & \cdots & \omega_{2m+1}^{2m+1} \end{pmatrix}.$$

A simple calculation reveals that $|\det \overline{N}(\infty)| = |\det U|^2 \neq 0$, where U is defined at (3.3).

Finally, using

$$e^{\beta_0}e^{\beta_1}\cdots e^{\beta_m} = \exp\left\{-\csc\left(\frac{\pi}{2m+2}\right)x^{1/(2m+2)}\right\}$$

we obtain

$$\begin{aligned}
&L_m(x) + (m+1)\log(2m+2) \\
&\quad = \log|e^{-\beta_0}e^{-\beta_1}\cdots e^{-\beta_m}\det\overline{N}(x)| \\
&\quad = \log\left|\exp\left\{\csc\left(\frac{\pi}{2m+2}\right)x^{1/(2m+2)}\right\}\det\overline{N}(x)\right| \\
&\quad = \csc\left(\frac{\pi}{2m+2}\right)x^{1/(2m+2)} + \log|\det\overline{N}(\infty)| + O(|e^{\beta_0}|) \\
&\quad = \csc\left(\frac{\pi}{2m+2}\right)x^{1/(2m+2)} + 2\log|\det U| + O\Bigg(\exp\left\{-\sin\left(\frac{\pi}{2m+2}\right)x^{1/(2m+2)}\right\}\Bigg),
\end{aligned}$$

as desired. □

Remark 3.3. From Lemma 3.2 we see that $L_m(x)/x^{1/(2m+2)}$ has a finite nonzero limit as $x \to \infty$. Also, it is easy to see that $L_m(x)/x$ has a finite nonzero limit $x \to 0+$. Therefore, the fundamental strip for the Mellin transform L_m^* of L_m is $\langle -1, -1/(2m+2)\rangle$.

For later purposes, it is important to note that $L_m^*(s)$ does not vanish for real $s \in \langle -1, -1/(2m+2)\rangle$. Indeed, $L_m(x) > 0$ for all $x > 0$, and hence $L_m^*(s) = \int_0^\infty L_m(x)x^{s-1}\,dx > 0$ for such s.

Remark 3.4. As a consequence of the Direct Mapping Theorem we see that L_m^* is continuable to a meromorphic function in the strip $\langle -1, \infty\rangle$, with

(3.4)
$$L_m^*(s) \asymp \left[-\frac{\csc\left(\frac{\pi}{2m+2}\right)}{s+\frac{1}{2m+2}}\right]_{s=-\frac{1}{2m+2}} + \left[\frac{-2\log|\det U| + (m+1)\log(2m+2)}{s}\right]_{s=0};$$

in particular, in this strip $L_m^*(s)$ has only simple poles, at $s = -1/(2m+2)$ and at $s = 0$.

Remark 3.5. Although our treatment keeps m fixed, the reader may be curious as to the large-m behavior of the term $2\log|\det U_m|$ appearing in Lemma 3.2. Using the well-known formula for the determinant of a Vandermonde matrix and approximating the double sum that results after taking logarithms by a double integral, we find

$$2\log|\det U_m| \sim m^2\left[\frac{1}{2}\log 2 + \iint_{0\le x<y\le 1}\log\sin\left(\frac{\pi}{2}(y-x)\right)\,dx\,dy\right] = -\frac{7\zeta(3)}{2\pi^2}m^2.$$

We omit the details.

Since $L_m(x) = \sum_{n=1}^{\infty} \log\left(1 + \frac{x}{\kappa_n(m)}\right)$, it follows from (2.4) and (2.10) that

$$L_m^*(s) = \frac{\pi}{s \sin(\pi s)} \sum_{n=1}^{\infty} \kappa_n^s(m), \tag{3.5}$$

or what is the same,

$$K_m(s) := \sum_{n=1}^{\infty} \kappa_n^s(m) = \frac{s \sin(\pi s)}{\pi} L_m^*(s). \tag{3.6}$$

Since, as discussed in the next subsection, $\kappa_n(m)$ grows like n^{2m+2}, (3.5)–(3.6) are valid (without continuation) in the strip $\langle -1, -1/(2m+2)\rangle$.

3.2 The generalized Dirichlet series $K_m(s)$

When we apply the Reverse Mapping Theorem to $L_{\mathbf{m}}^*$ in Section 3.3, we will need to verify the growth condition (2.8). This will rely on corresponding estimates for the one-dimensional L_m^*, and so we need to study the growth of $L_m^*(s)$, or equivalently of the generalized Dirichlet series $K_m(s)$.

We can do this using the following very sharp asymptotic estimate for $\kappa_n(m)$ (see also Gao et al. [6]).

Lemma 3.6 (Gao et al. [5], Theorem 2). *For $m = 0$ we have $\kappa_n(0) = \left[\left(n - \frac{1}{2}\right)\pi\right]^2$ for every n. For each fixed $m \ge 1$, as $n \to \infty$ we have*

$$\kappa_n(m) = \left[\left(n - \tfrac{1}{2}\right)\pi\right]^{2m+2}\left[1 + O\left(n^{-1}\exp\left\{-\pi \sin\left(\frac{\pi}{m+1}\right) n\right\}\right)\right]. \qquad \square$$

From Lemma 3.6 it is easily seen that the domain of absolute convergence for K_m is the strip $\langle -\infty, -1/(2m+2)\rangle$. In this strip we can write

$$K_m(s) = \widehat{K}_m(s) + \sum_{n=1}^{\infty}\left[\kappa_n^s(m) - \hat{\kappa}_n^s(m)\right]$$

where

$$\widehat{K}_m(s) := \sum_{n=1}^{\infty} \hat{\kappa}_n^s(m) = \left[\left(\tfrac{\pi}{2}\right)^{(2m+2)s} - \pi^{(2m+2)s}\right]\zeta(-(2m+2)s)$$

with $\hat{\kappa}_n(m) := \left[\left(n - \frac{1}{2}\right)\pi\right]^{2m+2}$; here ζ denotes Riemann's zeta function. Using Lemma 3.6 it is easy to check that the remainder series $\sum_{n=1}^{\infty}\left[\kappa_n^s(m) - \hat{\kappa}_n^s(m)\right]$ converges absolutely for all $s \in \mathbb{C}$ and defines an entire function of s that is $O(|s|)$ in any strip $\langle -R, R\rangle$ with $0 < R < \infty$. The generalized Dirichlet series $\widehat{K}_m(s)$ is meromorphically continuable for $s \in \mathbb{C}$, with a single simple pole at $s = -1/(2m+2)$. In any strip $\langle -R, R\rangle$, the continued $\widehat{K}_m(s)$ grows at most polynomially in $|s|$ (see, e.g., [16], Section 5.1). Putting the pieces of the above argument together, we have established the following result.

Lemma 3.7. *For any $0 < R < \infty$, the meromorphic continuation of the generalized Dirichlet series $K_m(s)$ at (3.6) grows at most polynomially in $|s|$ as $|s| \to \infty$ in the strip $\langle -R, R\rangle$.* $\square$

3.3 The general d-dimensional case: separation

With results regarding the one-dimensional processes in hand we now turn our attention to arbitrary dimension d and vector $\mathbf{m}$. The following simple lemma exploits the harmonic-sum structure (3.1) of $L^*_{\mathbf{m}}(s)$.

Lemma 3.8. *For any dimension $d \geq 1$,*

$$L^*_{\mathbf{m}}(s) = \left(\frac{s\sin(\pi s)}{\pi}\right)^{d-1} \prod_{j=1}^{d} L^*_{m_j}(s), \tag{3.7}$$

valid (without continuation) in the strip $\langle -1, -1/(2m_1+2)\rangle$.

Proof. When $d = 1$ the result is trivial (recalling Remark 3.3). For $d \geq 2$, we observe

$$L_{\mathbf{m}}(x) = \sum_{n_d=1}^{\infty} L_{(m_1,\dots,m_{d-1})}\left(\frac{x}{\kappa_{n_d}(m_d)}\right),$$

whence, by the separation property (2.10), (3.6), and induction,

$$\begin{aligned} L^*_{\mathbf{m}}(s) &= L^*_{(m_1,\dots,m_{d-1})}(s) \times \sum_{n_d=1}^{\infty} \kappa^s_{n_d}(m_d) = \frac{s\sin(\pi s)}{\pi} L^*_{(m_1,\dots,m_{d-1})}(s) L^*_{m_d}(s) \\ &= \left(\frac{s\sin(\pi s)}{\pi}\right)^{d-1} \prod_{j=1}^{d} L^*_{m_j}(s) \end{aligned}$$

for $s \in \langle -1, -1/(2m_1+2)\rangle$, as claimed. □

Remark 3.9. In preparation for our main theorem (Theorem 3.11) we next consider the singular expansion of the meromorphically continued $L^*_{\mathbf{m}}(s)$. In light of Lemma 3.8 it is enough to understand the meromorphically continued $L^*_{m_j}(s)$ for each j. Each $L^*_{m_j}(s)$ has a simple pole at $s = 0$, but $s\sin(\pi s)$ has a double zero at $s = 0$; thus, if $d \geq 2$, then, for each $j = 1, \dots, d$, $L^*_{\mathbf{m}}(s)$ may have a pole at $s = -1/(2m_j+2)$, but these are the only possible singularities. We say "possible" singularities since we have not considered whether $L^*_{m_j}(s)$ might vanish at $-1/(2m_k+2)$ for some $m_k > m_j$. However, when $m_k < m_j$, we know from Remark 3.3 that $L^*_{m_j}(-1/(2m_k+2)) \neq 0$.

Let

$$r_m(x) := L_m(x) - \left[\csc\left(\frac{\pi}{2m+2}\right) x^{1/(2m+2)} + 2\log|\det U| - (m+1)\log(2m+2)\right]$$

denote the exponentially small remainder term in the asymptotic expansion for $L_m(x)$ in Lemma 3.2. From Lemma 3.2 and (the proof of) the Direct Mapping Theorem, for $s \neq -1/(2m_j+2)$ we have [compare (3.4)]

$$L^*_{m_j}(s) = -\frac{\csc\left(\frac{\pi}{2m_j+2}\right)}{s + \frac{1}{2m_j+2}} + A_{m_j}(s). \tag{3.8}$$

Here

$$(3.9) \qquad A_{m_j}(s) = \int_0^1 L_{m_j}(x) x^{s-1}\,dx + \frac{-2\log|\det U_{m_j}| + (m_j+1)\log(2m_j+2)}{s} + \int_1^\infty r_{m_j}(x) x^{s-1}\,dx$$

is analytic in the strip $\langle -1, \infty \rangle$ except at $s = 0$.

Remark 3.10. Lemmas 3.7 and 3.8, together with the exponential growth of the sine function along vertical lines in $\mathbb{C}$, imply that $L^*_{\mathbf{m}}(s)$ satisfies the growth condition (2.8) for *every* $1 < r < \infty$.

3.4 Main theorem: Complete asymptotic expansion of $L_{\mathbf{m}}(x)$

With the previous results we are now able to obtain a complete asymptotic expansion for $L_{\mathbf{m}}(x)$. In order to state our main result, we suppose

$$m_1 = \cdots = m_{t_1} < m_{t_1+1} = \cdots m_{t_1+t_2} < \cdots < m_{t_1+\cdots+t_{g-1}+1} = \cdots = m_{t_1+\cdots+t_g},$$

where the number of groups of ties is $g \geq 1$ and, for $1 \leq \nu \leq g$, there is a tie of size $t_\nu \geq 1$ in group ν; thus $d = t_1 + \cdots + t_g$. We will denote the common m-value in the νth group by $\bar{m}_\nu$. Set

$$\xi_\nu := \frac{1}{2\bar{m}_\nu + 2}.$$

In order to use the Reverse Mapping Theorem, we need the singular expansion of $L^*_{\mathbf{m}}(s)$ near each pole $s = -\xi_\nu$ ($\nu = 1, \ldots, g$). From (3.7) and (3.8), near $s = -\xi_\nu$ we have

$$(3.10) \qquad L^*_{\mathbf{m}}(s) = \left(\frac{s\sin(\pi s)}{\pi}\right)^{d-1} \left(\prod_{n \neq \nu} \left[L^*_{\bar{m}_n}(s)\right]^{t_n}\right) \left(\frac{-\csc(\pi\xi_\nu)}{s+\xi_\nu} + A_{\bar{m}_\nu}(s)\right)^{t_\nu}.$$

The product of the first two of the three factors on the right in (3.10) is analytic for s near $-\xi_\nu$ and therefore has a Taylor expansion that can be computed up through the term of order $(s+\xi_\nu)^{t_\nu-1}$; in regard to this computation, note that repeated differentiation under the integrals in (3.9) is easily justified. Similarly, we can expand the analytic function $A_{\bar{m}_\nu}(s)$ up through order $(s+\xi_\nu)^{t_\nu-2}$ and use a multinomial expansion to obtain a Laurent series for the third factor up through order $(s+\xi_\nu)^{-1}$. Multiplying these expansions together we can get a Laurent series for (3.10) up through the term of order $(s+\xi_\nu)^{-1}$. Applying the Reverse Mapping Theorem (the growth condition is satisfied: see Remark 3.10) we arrive at our main result.

Theorem 3.11. *For any* $\mathbf{m}$*-integrated Brownian sheet in dimension* $d \geq 2$*,* $L_{\mathbf{m}}(x)$ *has as* $x \to \infty$ *a complete asymptotic expansion of the following form:*

$$(3.11) \qquad L_{\mathbf{m}}(x) = \sum_{\nu=1}^{g} \sum_{k=0}^{t_\nu-1} c_{\nu,k}\, x^{\xi_\nu} (\log x)^k + O(x^{-R})$$

for any $R > 0$*. The values of* $c_{\nu,k}$ *are computed as outlined in the previous paragraph.* □

Remark 3.12. In fact, the error term in Theorem 3.11 is exponentially small in a positive power of x. We have written a complete proof, but will provide only a sketch in the next paragraph (as the details are straightforward but laborious), that the remainder term is bounded by $\exp\left\{-cx^{1/(d(2m_d+2))}\right\}$ for a certain constant c depending on (d and) $\mathbf{m}$. We have not tried to optimize this bound; in particular, it may be that the power $1/(d(2m_d+2))$ can be improved to $1/[2(m_1+\cdots+m_d)+2d]$.

The Reverse Mapping Theorem 2.2 is proved by invoking the inverse Mellin transform formula (2.5) with $c=-\eta$ and then shifting the line of integration rightward to $\operatorname{Re} s=-\gamma$ by means of the residue theorem; the growth condition (2.8) is used to justify this shift rigorously. One then uses the growth condition again to bound the shifted integral by $O(x^{\gamma})$. When sharper growth estimates of $f^*(s)$ are available, as they are in our case $f=L_{\mathbf{m}}$ (cf. Remark 3.10), one can let $|\gamma|$ grow with x and obtain sharper remainder bounds. Our proof for $L_{\mathbf{m}}$ uses well-known growth estimates for the Riemann zeta function in vertical strips and takes $\gamma(x)=-\tilde{c}\,x^{1/(d(2m_d+2))}+O(1)$ for a suitable positive constant $\tilde{c}$.

Because $h(x)=\frac{1}{2}L_{\mathbf{m}}(2x)$, it follows from Theorem 3.11 that $h(x)$ also has an expansion of the form (3.11), namely, for any $R>0$,

$$h(x)=\sum_{\nu=1}^{g}\sum_{k=0}^{t_\nu-1} c_{\nu,k}(0)\,x^{\xi_\nu}(\log x)^k+O(x^{-R}) \tag{3.12}$$

where

$$c_{\nu,k}(0):=2^{\xi_\nu-1}\sum_{\ell=k}^{t_\nu-1} c_{\nu,\ell}\binom{\ell}{k}(\log 2)^{\ell-k}$$

(and error term as refined in Remark 3.12). In order to apply Sytaja's theorem in Section 5, we will need the following extension of Theorem 3.11 to the derivatives of h.

Lemma 3.13. *For $j=0,1,2,\dots,$ the function $h^{(j)}$ has as $x\to\infty$ a complete asymptotic expansion of the form*

$$h^{(j)}(x)=\sum_{\nu=1}^{g}\sum_{k=0}^{t_\nu-1} c_{\nu,k}(j)\,x^{\xi_\nu-j}(\log x)^k+O(x^{-(R+j)}) \tag{3.13}$$

for any $0<R<\infty$. The expansions are obtained by successive termwise differentiations, starting with (3.12). In particular, with $c_{\nu,k}(j):=0$ whenever $k\geq t_\nu$,

$$\begin{aligned}
c_{\nu,k}(1)&=\xi_\nu c_{\nu,k}(0)+(k+1)c_{\nu,k+1}(0),\\
c_{\nu,k}(2)&=(\xi_\nu-1)c_{\nu,k}(1)+(k+1)c_{\nu,k+1}(1)\\
&=\xi_\nu(\xi_\nu-1)c_{\nu,k}(0)+(2\xi_\nu-1)(k+1)c_{\nu,k+1}(0)+(k+2)(k+1)c_{\nu,k+2}(0),
\end{aligned}$$

and, for any $0<R<\infty$,

$$h(x)-xh'(x)=\sum_{\nu=1}^{g}\sum_{k=0}^{t_\nu-1}[(1-\xi_\nu)c_{\nu,k}(0)-(k+1)c_{\nu,k+1}(0)]x^{\xi_\nu}(\log x)^k+O(x^{-R}).$$

Proof. Denote the eigenvalues of $\mathcal{A}_{\mathbf{m}}$ by $a_{\mathbf{n}}$, as at (1.5). Then, for all $x \geq 0$, $h(x) = \frac{1}{2}\sum_{\mathbf{n}} \log(1 + 2a_{\mathbf{n}}x)$ and the dominated convergence theorem justifies the termwise differentiation giving, for $j = 1, 2, \ldots$ and $x \geq 0$,

$$h^{(j)}(x) = \tfrac{1}{2}\,(-1)^{j-1}(j-1)!\sum_{\mathbf{n}}(2a_{\mathbf{n}})^j(1 + 2a_{\mathbf{n}}x)^{-j}.$$

In particular, each function $h^{(j)}$ is monotone; and as $x \to 0+$ we have the simple estimates

$$h(x) = (1 + o(1))\,x\sum_{\mathbf{n}} a_{\mathbf{n}}, \qquad h^{(j)}(x) = (1 + o(1))\,\tfrac{1}{2}\,(-1)^{j-1}(j-1)!\sum_{\mathbf{n}}(2a_{\mathbf{n}})^j.$$

It is then easy to use induction on j, together with Corollary 2.3 and Remark 3.10, to complete the proof. □

Remark 3.14. For each $j = 0, 1, 2, \ldots$, the error term in (3.13) is exponentially small in a positive power of x; compare Remark 3.12.

3.5 Expansion of $L_{\mathbf{m}}(x)$: examples

We next give three simple examples of the computations entering into the expansion in Theorem 3.11.

3.5.1 Lead-order asymptotics in the general d-dimensional case ($d \geq 2$)

Suppose $d \geq 2$ and

$$m := m_1 = \cdots = m_t < m_{t+1} \leq \cdots \leq m_d.$$

According to Theorem 3.11, the lead-order asymptotics for $L_{\mathbf{m}}(x)$ are given by

$$L_{\mathbf{m}}(x) = (1 + o(1))\,c_{1,t-1}\,x^{1/(2m+2)}(\log x)^{t-1},$$

since [from (3.15) below; see also Remark 3.9] $c_{1,t-1} > 0$. This lead-order term corresponds via the Reverse Mapping Theorem to the term of order $(s+\xi_1)^{-t}$ in the expansion of $L^*_{\mathbf{m}}(s)$ near its pole at $s = -\xi_1$. From (3.10) we see immediately that, as $s \to -\xi_1$,

$$\begin{aligned} L^*_{\mathbf{m}}(s) &= (1 + o(1))\left(\frac{\xi_1\sin(\pi\xi_1)}{\pi}\right)^{d-1}\left(\prod_{j=t+1}^{d} L^*_{m_j}(-\xi_1)\right)\left(\frac{-\csc(\pi\xi_1)}{s+\xi_1}\right)^{t} \\ &= -(1 + o(1))\frac{\left[\sin\left(\frac{\pi}{2m+2}\right)\right]^{d-1-t}}{[(2m+2)\pi]^{d-1}(t-1)!}\left(\prod_{j=t+1}^{d} L^*_{m_j}\left(-\frac{1}{2m+2}\right)\right)\frac{(-1)^{t-1}(t-1)!}{(s+\xi_1)^t} \end{aligned}$$

from which it follows that

$$c_{1,t-1} = \frac{\left[\sin\left(\frac{\pi}{2m+2}\right)\right]^{d-1-t}}{[(2m+2)\pi]^{d-1}(t-1)!}\left(\prod_{j=t+1}^{d} L^*_{m_j}\left(-\frac{1}{2m+2}\right)\right). \tag{3.14}$$

Thus, in general, computation of the lead-order term for $L_{\mathbf{m}}(x)$ requires (numerical) evaluation of the fundamental-strip Mellin transform values

$$L^*_{m_j}\left(-\frac{1}{2m+2}\right) = \int_0^\infty L_{m_j}(x)\, x^{-(2m+3)/(2m+2)}\, dx > 0. \tag{3.15}$$

However, in the particular equal-m's case where $t = d$ we obtain the simpler result

$$L_{\mathbf{m}}(x) = (1+o(1)) \frac{\csc(\frac{\pi}{2m+2})}{[(2m+2)\pi]^{d-1}(d-1)!} x^{1/(2m+2)} (\log x)^{d-1}.$$

Remark 3.15. From (3.6) we see that an alternative to numerical evaluation of the integral (3.15) is numerical evaluation of the generalized Dirichlet series $K_{m_j}\left(-\frac{1}{2m+2}\right)$. Evaluation of such sums arises in the approach of [9], which is more computationally intensive since it involves numerical root-finding.

3.5.2 Complete asymptotic expansion: distinct m_j's

Given Theorem 3.11, it is almost trivial to obtain a full asymptotic expansion for $L_{\mathbf{m}}(x)$ when the vector $\mathbf{m}$ has distinct components. That is, suppose that $g = d \geq 2$, so that $t_\nu = 1$ for $\nu = 1, \ldots, d$. Then Theorem 3.11 implies

$$L_{\mathbf{m}}(x) = \sum_{\nu=1}^{d} C_\nu\, x^{1/(2m_\nu+2)} + O(x^{-R}) \tag{3.16}$$

for any $0 < R < \infty$, where

$$C_\nu := \frac{\left[\sin\left(\frac{\pi}{2m_\nu+2}\right)\right]^{d-2} \prod_{j\neq\nu} L^*_{m_j}\left(-\frac{1}{2m_\nu+2}\right)}{[(2m_\nu+2)\pi]^{d-1}}. \tag{3.17}$$

3.5.3 Full asymptotic expansion: $d = 2$

Theorem 3.11 provides a complete asymptotic expansion of $L_{\mathbf{m}}(x)$ for any vector $\mathbf{m}$. However, calculations quickly become quite cumbersome to perform by hand; symbolic-manipulation software such as `Mathematica` or `Maple` can then be of great help. Complementing the above result for distinct m's, here is another case where we can spell out the expansion without too much notation.

Suppose that $m := m_1 = m_2$, i.e., that $g = 1$ and $d = t_1 = 2$. We then find

$$L_{\mathbf{m}}(x) = c_{1,1}\, x^{1/(2m+2)} \log x + c_{1,0}\, x^{1/(2m+2)} + O(x^{-R})$$

for any $0 < R < \infty$, where

$$c_{1,1} = \frac{\csc\left(\frac{\pi}{2m+2}\right)}{(2m+2)\pi}, \tag{3.18}$$

$$c_{1,0} = \frac{A_m\left(-\frac{1}{2m+2}\right)}{(m+1)\pi} + \frac{\csc\left(\frac{\pi}{2m+2}\right)}{\pi} + \frac{\cos\left(\frac{\pi}{2m+2}\right)\csc^2\left(\frac{\pi}{2m+2}\right)}{2m+2}, \tag{3.19}$$

and $A_m(\cdot)$ is computed via (3.9). When $m = 0$ in this example, i.e., when V is the L^2-norm of *two-dimensional Brownian sheet*, we have from Lemma 3.1 that

$$L_0(x) \equiv \log\cosh(x^{1/2})$$

and

$$A_0(-1/2) = \int_0^1 L_0(x)\, x^{-3/2}\, dx - 2\log 2 + \int_1^\infty \Big[L_0(x) - x^{1/2} + \log 2\Big]\, x^{-3/2}\, dx \approx -0.3624.$$

Thus,

$$L_{(0,0)}(x) = \frac{1}{2\pi} x^{1/2}\log x + c_{1,0}\, x^{1/2} + O(x^{-R})$$

for any $0 < R < \infty$, and here

$$c_{1,0} = \frac{1 + A_0(-1/2)}{\pi} \approx 0.2029.$$

4 Reversion of $h'(x) = \varepsilon$

In principle we are now in position to use Sytaja's theorem in conjunction with the asymptotic expansions for $h(x) - xh'(x)$ and $h''(x)$ in Lemma 3.13 to state a strong small-deviations result for integrated Brownian sheets. The problem is that, for more explicit results, we still must solve $h'(x) = \varepsilon$ to get $x^* \equiv x^*(\varepsilon)$ in order to obtain the needed expansions in ε, up to an additive term $o(1)$ for $h(x^*) - \varepsilon\, x^*$ and up to a factor $1 + o(1)$ for $(x^*)^2\, h''(x^*)$.

In this Section 4 we discuss the reversion of the asymptotic expansion for $h'(x)$ given in Lemma 3.13. We begin in Section 4.1 by showing that the exponentially small remainders in all our asymptotic expansions may safely be ignored when carrying out reversion and applying Sytaja's theorem. In Section 4.2 we determine the lead-order asymptotics for x^*. Clearly, the lead-order asymptotics agree for x^* and the (unique real) solution $x_0 \equiv x_0(\varepsilon)$ to the equation

$$\varepsilon = \sum_{k=0}^{t_1 - 1} c_{1,k}(1)\, x^{\xi_1 - 1} (\log x)^k \tag{4.1}$$

obtained from the equation $\varepsilon = h'(x)$ by including only those terms with $\nu = 1$ (corresponding to the largest power of x) in the expression (3.13) (with $j = 1$) for $h'(x)$. Next, in Section 4.3 we obtain a complete asymptotic expansion for x^* in terms of elementary functions and x_0. Finally, in Section 4.4 we discuss exact computation of x_0.

4.1 Truncating the asymptotic expansion for h' suffices

Our next result implies that, as far as obtaining lead-order asymptotics for the small-ball probability, we may act as if the asymptotic expansions in Lemma 3.13 were exact expressions.

Lemma 4.1. *Let $\hat{h}$ denote the truncated asymptotic expansion for h:*

$$\hat{h}(x) := \sum_{\nu=1}^{g} \sum_{k=0}^{t_\nu - 1} c_{\nu,k}(0)\, x^{\xi_\nu} (\log x)^k,$$

and let $\hat{x}$ satisfy $\hat{h}'(\hat{x}) = \varepsilon$. As $\varepsilon \to 0+$, the following errors all tend to zero faster than any power of ε:

$$\hat{x} - x^*, \quad \hat{h}^{(j)}(\hat{x}) - h^{(j)}(x^*) \text{ (for } j = 0, 1, \dots\text{)}, \quad [\hat{h}(\hat{x}) - \hat{x}\hat{h}'(\hat{x})] - [h(x^*) - \varepsilon\, x^*].$$

Observe that $\hat{h}'(x)$ decreases monotonically to 0 for sufficiently large x; thus for sufficiently small ε there exists a unique solution $\hat{x} \equiv \hat{x}(\varepsilon)$ to $\hat{h}'(\hat{x}) = \varepsilon$. We remark in passing that $\hat{h}^{(j)}$ can also be obtained by truncating the asymptotic expansion for $h^{(j)}$ given in Lemma 3.13.

Proof. From definitions and Lemma 3.13 we have

$$\hat{h}'(\hat{x}) = \varepsilon = h'(x^*) = \hat{h}'(x^*) + O((x^*)^{-R}) \text{ for any } 0 < R < \infty,$$

which by the simple Lemma 4.3 below can be written

$$\hat{h}'(\hat{x}) - \hat{h}'(x^*) = O(\varepsilon^R) \text{ for any } 0 < R < \infty,$$

as $\varepsilon \to 0+$. It follows using the mean value theorem that

$$\hat{x} - x^* = O(\varepsilon^R) \text{ for any } 0 < R < \infty.$$

The other assertions follow readily. □

Remark 4.2. The errors in Lemma 4.1 are each exponentially small in a positive power of $1/\varepsilon$. This assertion follows easily from the case $j = 1$ of Remark 3.14.

4.2 Lead-order reversion in the general d-dimensional case ($d \geq 2$)

If all we want is a standard "weak" small-deviations result, we need only obtain the lead-order asymptotics for $x^*(\varepsilon)$ [equivalently, for $x_0(\varepsilon)$]. This is easy. Recall from Lemma 3.13 that $h'(x) = (1 + o(1))\, c_{1,t-1}(1)\, x^{\xi-1} (\log x)^{t-1}$, where $t := t_1$ is the number of m_j's equal to the smallest value $m := m_1$ and $\xi := \xi_1 = 1/(2m+2)$. The coefficient $c_{1,t-1}(1)$ is given by $c_{1,t-1}(1) = \xi 2^{\xi-1} c_{1,t-1}$, with $c_{1,t-1}$ given by (3.14). The proof of the following lemma is left to the reader.

Lemma 4.3. *If $h'(x^*) = \varepsilon$ and x_0 is the solution to* (4.1)*, then as $\varepsilon \to 0+$ we have $x^* = (1 + o(1))\, x_0 = (1 + o(1))\, \tilde{x}_0$, where*

$$\tilde{x}_0 \equiv \tilde{x}_0(\varepsilon) := \left[\frac{c_{1,t-1}(1)}{(1-\xi)^{t-1}} \cdot \frac{1}{\varepsilon} \left(\log \frac{1}{\varepsilon} \right)^{t-1} \right]^{\frac{1}{1-\xi}}. \tag{4.2}$$ □

We will use Lemma 4.3 in Section 5 to obtain a rather simple "weak" small-deviations result for arbitrary $\mathbf{m}$; see Theorem 5.1. Lemma 4.3 also provides us with all the asymptotic information about $h''(x^*)$ we need in applying Sytaja's theorem:

Corollary 4.4. *If $h'(x^*) = \varepsilon$, then, with $\tilde{x}_0$ defined at (4.2), as $\varepsilon \to 0+$ we have*

$$-(x^*)^2 h''(x^*) = (1 + o(1))\,(1 - \xi)\,\tilde{x}_0\,\varepsilon$$

Proof. This is routine. Recall Lemma 3.13 and in particular the coefficients

$$c_{1,t-1}(2) = \xi(\xi - 1)2^{\xi-1} c_{1,t-1} = -(1 - \xi) c_{1,t-1}(1).$$

Then

$$\begin{aligned} -(x^*)^2 h''(x^*) &= -(1 + o(1))\,\tilde{x}_0^2\,h''(\tilde{x}_0) = -(1 + o(1))\,c_{1,t-1}(2)\,\tilde{x}_0^{\xi}\,(\log \tilde{x}_0)^{t-1} \\ &= -(1 + o(1))\,c_{1,t-1}(2)\,\tilde{x}_0^{\xi} \left(\frac{1}{1-\xi} \log \frac{1}{\varepsilon} \right)^{t-1}, \end{aligned}$$

and it is easy to check from definitions that

$$-c_{1,t-1}(2)\,\tilde{x}_0^{\xi} \left(\frac{1}{1-\xi} \log \frac{1}{\varepsilon} \right)^{t-1} = (1 - \xi)\tilde{x}_0 \varepsilon. \qquad \square$$

4.3 Reversion: an asymptotic expansion for x^*

We next derive a complete asymptotic expansion for x^* in terms of elementary functions and the solution x_0 to (4.1). We will discuss exact computation of x_0 in Section 4.4.

Dropping the error term from the expansion (3.13) (with $j = 1$) for $h'(x)$ (as justified by Lemma 4.1), let us write the equation $\varepsilon = h'(x)$ in the form

$$\varepsilon = \sum_{\nu=1}^{g} f_\nu(x), \tag{4.3}$$

where

$$f_\nu(x) := x^{-\eta_\nu} \sum_{k=0}^{t_\nu - 1} a_{\nu,k} (\log x)^k, \quad \nu = 1, \dots, g; \tag{4.4}$$

here, for abbreviation, we have set $a_{\nu,k} := c_{\nu,k}(1)$ and $\eta_\nu := 1 - \xi_\nu = (2\bar{m}_\nu + 1)/(2\bar{m}_\nu + 2)$, so that

$$1/2 \le \eta_1 < \eta_2 < \cdots \eta_g < 1.$$

(The notation $\bar{m}_\nu$ is as in Section 3.4.) If $g = 1$, then we have simply $x^* = x_0$; so we assume now that $g \ge 2$.

The main result of this subsection is the following complete asymptotic expansion for x^*/x_0 in terms of inductively defined quantities y_j ($j = 0, 1, 2, \dots$). Let $y_0 := 1$. Suppose that $j \ge 1$ and that we have defined $y_0, \dots, y_{j-1}$. Then set

$$y_{j-1}^{+} := y_0 + y_1 + \cdots + y_{j-1} \quad \text{and} \quad x_{j-1} := x_0\, y_{j-1}^{+}$$

and define

$$y_j := \frac{y_{j-1}^{+}\,[\sum_{\nu=1}^{g} f_\nu(x_{j-1}) - \varepsilon]}{\eta_1 f_1(x_{j-1}) - x_{j-1}^{-\eta_1} \sum_{k=0}^{t_1-1} k a_{1,k} (\log x_{j-1})^{k-1}}. \tag{4.5}$$

Proposition 4.5. *For each* $j = 0, 1, \dots$ *we have*

$$\frac{x^*}{x_0} = 1 + y_1 + y_2 + \cdots + (1 + o(1)) y_j \tag{4.6}$$

and

$$y_j = O\left(\left(\varepsilon^{\frac{\eta_2}{\eta_1} - 1} \left(\log \frac{1}{\varepsilon} \right)^{\Delta} \right)^{j} \right), \tag{4.7}$$

where $\Delta := \max\{t_\nu - 1 - (t_1 - 1)\frac{\eta_\nu}{\eta_1} : 2 \le \nu \le g\}$.

Note that, with $x_{-1} := 0$, (4.6) can be written equivalently as

$$x^* = x_{j-1} + (1 + o(1)) x_0 y_j \quad (j = 0, 1, \dots). \tag{4.8}$$

Before we prove the proposition, we illustrate its application in two special cases.

Example 4.6 ($t_1 = 1$). Suppose $t_1 = 1$, whence $f_1(x)$ reduces to $a_{1,0} x^{-\eta_1}$ and it is elementary that $x_0 = (a_{1,0}/\varepsilon)^{1/\eta_1}$. In this case, (4.5) simplifies to

$$y_j = \frac{y_{j-1}^{+}}{\eta_1 a_{1,0}} x_{j-1}^{\eta_1} \left[\sum_{\nu=1}^{g} f_\nu(x_{j-1}) - \varepsilon \right]. \tag{4.9}$$

If we assume further, as in Section 3.5.2, that all the m_j's are distinct (i.e., $g = d$), then $f_\nu(x)$ simplifies to $a_{\nu,0}\, x^{-\eta_\nu}$ and (4.9) can be written in the form

$$y_j = \frac{(y_{j-1}^{+})^{1+\eta_1}}{\eta_1} \left[\sum_{\nu=1}^{d} a_{\nu,0}\, a_{1,0}^{-\eta_\nu/\eta_1} \varepsilon^{(\eta_\nu/\eta_1)-1} (y_{j-1}^{+})^{-\eta_\nu} - 1 \right]. \tag{4.10}$$

From this it is easy to prove by induction that each y_j has an asymptotic expansion in increasing powers of ε wherein each power is a nonnegative integer combination of the numbers $(\eta_\nu/\eta_1) - 1$ with $2 \le \nu \le d$. The same is therefore true of x^*/x_0.

Example 4.7 ($m_1 = 0$). Suppose $m_1 = \bar{m}_1 = 0$, whence $\eta_1 = 1/2$; since $\bar{m}_2 \ge 1$, we also have $\eta_2 \ge 3/4$. In this case it is not hard to see that the finite expansion

$$\begin{aligned} \frac{x^*}{x_0} &= 1 + y_1 + y_2 + O(y_3) = 1 + y_1 + y_2 + O\left(\varepsilon^{6\eta_2 - 3} \left(\log \frac{1}{\varepsilon} \right)^{3\Delta} \right) \\ &= 1 + y_1 + y_2 + O\left(\varepsilon^{3/2} \left(\log \frac{1}{\varepsilon} \right)^{3\Delta} \right) \end{aligned}$$

is sufficient to obtain the lead-order asymptotics for the small-ball probability. To be concrete, suppose further that there are no ties, so that we are in the context of (4.10) and $x_0 = (a_{1,0}/\varepsilon)^2$. Then

$$y_1 = 2\sum_{\nu=2}^{d} a_{\nu,0}\, a_{1,0}^{-2\eta_\nu}\, \varepsilon^{2\eta_\nu - 1},$$

$$y_2 = -4\left[\sum_{\nu=2}^{d} a_{\nu,0}\, a_{1,0}^{-2\eta_\nu}\, \varepsilon^{2\eta_\nu-1}\right]\left[\sum_{\nu=2}^{d}\left(\eta_\nu - \frac{3}{4}\right) a_{\nu,0}\, a_{1,0}^{-2\eta_\nu}\, \varepsilon^{2\eta_\nu-1}\right] + O(\varepsilon^{6\eta_2-3})$$

$$= -4\left(\eta_2 - \frac{3}{4}\right) a_{2,0}^2\, a_{1,0}^{-4\eta_2}\, \varepsilon^{4\eta_2-2} + O(\varepsilon^{2(\eta_2+\eta_3)-2}) + O(\varepsilon^{6\eta_2-3}), \tag{4.11}$$

where the first remainder term in (4.11) is used if $d \ge 3$, and the second if $d = 2$. So, whether $\eta_2 > 3/4$ or $\eta_2 = 3/4$, and whether $d \ge 3$ or $d = 2$,

$$\frac{x^*}{x_0} = 1 + y_1 + o(\varepsilon) = 1 + 2\sum_{\nu=2}^{d} a_{\nu,0}\, a_{1,0}^{-2\eta_\nu}\, \varepsilon^{2\eta_\nu-1} + o(\varepsilon),$$

which we will see in Example 5.8 is sufficient to obtain the lead-order asymptotics for the small-ball probability.

Proof of Proposition 4.5. We prove (4.8) and (4.7) together by induction on j. The base case $j = 0$ of the induction is simple. For $j \ge 1$, our induction hypothesis is that the $(j-1)$st instances of (4.8) and (4.7) hold; in particular, we can write

$$x^* = x_{j-2} + x_0 y_{j-1} + x_0 y = x_{j-1} + x_0 y = x_{j-1}\left(1 + \frac{y}{y_{j-1}^+}\right)$$

for some $y = o(y_{j-1})$. For $\nu = 1, \ldots, g$ we then have, from (4.4),

$$f_\nu(x^*) = f_\nu\left(x_{j-1}\left(1 + \frac{y}{y_{j-1}^+}\right)\right)$$

$$= x_{j-1}^{-\eta_\nu}\left(1 + \frac{y}{y_{j-1}^+}\right)^{-\eta_\nu} \sum_{k=0}^{t_\nu-1} a_{\nu,k}\left[\log x_{j-1} + \log\left(1 + \frac{y}{y_{j-1}^+}\right)\right]^k.$$

For $\nu \ge 2$ we conclude

$$f_\nu(x^*) - f_\nu(x_{j-1}) = O(y\, f_\nu(x_{j-1})) = O(y\, x_0^{-\eta_\nu}\, (\log x_0)^{t_\nu - 1})$$

$$= O\left(y\, \varepsilon^{\eta_\nu/\eta_1}\left(\log\frac{1}{\varepsilon}\right)^{t_\nu-1-(t_1-1)(\eta_\nu/\eta_1)}\right) = O\left(y\, \varepsilon^{\eta_2/\eta_1}\left(\log\frac{1}{\varepsilon}\right)^{\Delta}\right), \tag{4.12}$$

where the third equality follows from Lemma 4.3. For $\nu = 1$ we find

$$f_1(x^*) - f_1(x_{j-1}) = -(1+o(1))\frac{y}{y_{j-1}^+}\left[\eta_1 f_1(x_{j-1}) - x_{j-1}^{-\eta_1}\sum_{k=0}^{t_1-1} k a_{1,k} (\log x_{j-1})^{k-1}\right], \tag{4.13}$$

which we note is of the same order as $y\varepsilon$. Therefore,

$$\varepsilon = \sum_{\nu=1}^{g} f_\nu(x^*) = \sum_{\nu=1}^{g} f_\nu(x_{j-1}) - (1+o(1))\frac{y}{y_{j-1}^{+}}\left[\eta_1 f_1(x_{j-1}) - x_{j-1}^{-\eta_1}\sum_{k=0}^{t_1-1} k a_{1,k}(\log x_{j-1})^{k-1}\right],$$

and so [recalling (4.5)] $y = (1+o(1))y_j$. This establishes the jth instance of (4.8).

To establish the jth instance of (4.7), we note again that the denominator of (4.5) is precisely of order ε. We must also estimate the difference appearing in the numerator. Suppose that $j \geq 2$; the estimation for $j = 1$ is similar and left to the reader. By slight modifications of the arguments for (4.12)–(4.13), for $\nu \geq 2$ we have

$$f_\nu(x_{j-1}) - f_\nu(x_{j-2}) = O\left(y_{j-1}\,\varepsilon^{\eta_2/\eta_1}\left(\log\frac{1}{\varepsilon}\right)^{\Delta}\right), \tag{4.14}$$

and, also utilizing (4.5), for $\nu = 1$ we have

$$\begin{aligned} f_1(x_{j-1}) - f_1(x_{j-2}) &= -\frac{y_{j-1}}{y_{j-2}^{+}}\left[\eta_1 f_1(x_{j-2}) - x_{j-2}^{-\eta_1}\sum_{k=0}^{t_1-1} k a_{1,k}(\log x_{j-2})^{k-1}\right] + O\big(y_{j-1}^2\varepsilon\big) \\ &= -\left[\sum_{\nu=1}^{g} f_\nu(x_{j-2}) - \varepsilon\right] + O\big(y_{j-1}^2\varepsilon\big)\,. \end{aligned} \tag{4.15}$$

Summing the g equations (4.14)–(4.15) and rearranging,

$$\sum_{\nu=1}^{g} f_\nu(x_{j-1}) - \varepsilon = O\left(y_{j-1}\,\varepsilon^{\eta_2/\eta_1}\left(\log\frac{1}{\varepsilon}\right)^{\Delta}\right) + O\big(y_{j-1}^2\varepsilon\big)\,.$$

By the induction hypothesis and our assumption that $j \geq 2$, the first of the two $O(\cdot)$ terms predominates and the estimate

$$\sum_{\nu=1}^{g} f_\nu(x_{j-1}) - \varepsilon = O\left(\varepsilon\left(\varepsilon^{\frac{\eta_2}{\eta_1}-1}\left(\log\frac{1}{\varepsilon}\right)^{\Delta}\right)^{j}\right)$$

is established, completing the induction. □

4.4 Computation of x_0

There remains the task of solving the following equation to obtain x_0:

$$\varepsilon = f_1(x) = \sum_{k=0}^{t-1} a_k\, x^{-\eta}(\log x)^k, \tag{4.16}$$

with $t := t_1$, $a_k := a_{1,k} = c_{1,k}(1)$, and $\eta := \eta_1 = 1 - \xi_1 = (2m_1+1)/(2m_1+2)$. As has already been noted in Example 4.6, this is trivial if $t = 1$: then $x_0 = (a_0/\varepsilon)^{1/\eta}$. For general t, of course, one can resort (for given $\varepsilon > 0$) to Newton's method or other root-finding methods. What we will derive in this subsection is a series representation of the solution, from which will follow an asymptotic expansion for x_0 sufficient to yield a complete asymptotic expansion for the logarithm of the small-ball probability, but *not* for that probability itself (for which the exact value of x_0 should be used).

It is instructive to begin by considering the case $t = 2$. Setting $w := -\eta[\log x + (a_0/a_1)]$, equation (4.16) can be rewritten as

$$we^w = -\frac{\eta}{a_1} \exp\{-\eta a_0/a_1\}\varepsilon.$$

This equation for w cannot be solved in terms of elementary functions, but it does have the solution

$$w = W\left(-\frac{\eta}{a_1} \exp\{-\eta a_0/a_1\}\varepsilon\right),$$

in terms of the Lambert W-function. (See [2]; more specifically, W here is the branch W_{-1} in the notation there. We could alternatively work in terms of the closely related glog function defined in [8].) According to the paragraph following equation (4.20) in [2] we have, in terms of $z := (\eta/a_1)\exp\{-\eta a_0/a_1\}\varepsilon$, the expansion

$$w = W(-z) = \log z - \log\log\frac{1}{z} + \sum_{r=0}^{\infty}\sum_{s=1}^{\infty} d_{rs}\left(\log\log\frac{1}{z}\right)^s (\log z)^{-(r+s)} \tag{4.17}$$

with d_{rs} given in terms of Stirling numbers of the first kind (see equation 1.2.9-(26) in [11]) by $d_{rs} = \frac{1}{s!}(-1)^r \begin{bmatrix} r+s \\ r+1 \end{bmatrix}$. As remarked in [2], the series in (4.17) is absolutely and uniformly convergent for small ε and also serves as an asymptotic expansion for $w - \log z + \log\log\frac{1}{z}$ as $\varepsilon \to 0+$. It is not hard to show that this result can be rearranged to one of the following form, wherein the series has the same convergence and asymptotic expansion properties as in (4.17):

$$x_0 = \tilde{x}_0 \times \left[1 + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \tilde{d}_{rs}\left(\log\log\frac{1}{\varepsilon}\right)^s \left(\log\frac{1}{\varepsilon}\right)^{-(r+s)}\right] \quad \text{with } \tilde{d}_{00} = 0. \tag{4.18}$$

Recall that, in general,

$$\tilde{x}_0 := \left[\frac{a_{t-1}}{\eta^{t-1}} \cdot \frac{1}{\varepsilon} \cdot \left(\log\frac{1}{\varepsilon}\right)^{t-1}\right]^{1/\eta} \tag{4.19}$$

is the lead-order approximant to x_0 found in Lemma 4.3, and for $t = 2$ this reduces to $\tilde{x}_0 = \left(\frac{a_1}{\eta} \cdot \frac{1}{\varepsilon} \cdot \log\frac{1}{\varepsilon}\right)^{1/\eta}$. In particular, the first correction term in (4.18) then has coefficient $\tilde{d}_{01} = d_{01}/\eta = 1/\eta$.

The expansion (4.18) can be extended to general values of t:

Lemma 4.8. *For general t, the solution x_0 to* (4.16) *has an expansion of the form* (4.18)–(4.19). *The series in* (4.18) *is absolutely and uniformly convergent for small $\varepsilon > 0$, and when rearranged to*

$$\sum_{\ell=1}^{\infty}\sum_{s=0}^{\ell} \tilde{d}_{\ell-s,s}\left(\log\log\frac{1}{\varepsilon}\right)^s \left(\log\frac{1}{\varepsilon}\right)^{-\ell} \tag{4.20}$$

provides a complete asymptotic expansion for $(x_0/\tilde{x}_0) - 1$*. The lead term in* (4.20) *has coefficient* $\tilde{d}_{01} = (t-1)^2/\eta$*.*

Proof. We give only a sketch of the proof. We can write the result of Lemma 4.3 in the form

$$w := -\eta \log x_0 = -\sigma^{-1} - (t-1)\sigma^{-1}\tau - \lambda - v, \tag{4.21}$$

with $\lambda := \log(a_{t-1}/\eta^{t-1})$, where $v = o(1)$ and

$$\sigma := \frac{1}{\log(1/\varepsilon)} \quad \text{and} \quad \tau := \frac{\log\log(1/\varepsilon)}{\log(1/\varepsilon)}.$$

Substituting (4.21) into the defining equation (4.16) for x_0, we find

$$\varepsilon = e^w \sum_{k=0}^{t-1} a_k \, (-w/\eta)^k = e^w \sum_{k=0}^{t-1} a_k \, \eta^{-k} \left[\sigma^{-1} + (t-1)\sigma^{-1}\tau + \lambda + v\right]^k,$$

or equivalently that v is a root of the function

$$F(\zeta) := e^{\zeta} - 1 - \left\{[1 + (t-1)\tau + \lambda\sigma + \sigma\zeta]^{t-1} - 1\right\} \\ - e^{-\lambda}\sigma^{t-1} \sum_{k=0}^{t-2} a_k \, \eta^{-k}\sigma^{-k} \, [1 + (t-1)\tau + \lambda\sigma + \sigma\zeta]^k .$$

The proof now proceeds as on pp. 347–349 in [2] to obtain a series representation for v, and then we exponentiate to obtain (4.18). We omit the details and will be content here to verify the asserted value of $\tilde{d}_{01}$. Indeed, our proof sketch shows that $\eta \, \tilde{d}_{01}$ equals

$$\frac{1}{2\pi i} \int_{|\zeta| = \pi} G(\zeta) \, d\zeta,$$

where the integration is taken counterclockwise and $G(\zeta)$ is the coefficient of $\sigma^0\tau^1$ in $\zeta F'(\zeta)/F(\zeta)$, namely,

$$G(\zeta) = (t-1)^2 \zeta e^{\zeta} \left(e^{\zeta} - 1\right)^{-2}.$$

Evaluating the integral completes the proof. □

Remark 4.9. In Section 5 we will see that, when viewed as an asymptotic expansion for x_0, (4.18) is not accurate enough to use in obtaining a strong small-deviations result but is good enough (see Theorem 5.3) to yield an expansion for the logarithm of the small-ball probability.

5 Small deviation estimates

We now have at hand all of the ingredients we need to apply Sytaja's theorem and obtain $p \equiv p(\varepsilon) := \mathbf{P}(V^2 \le \varepsilon)$ up to a factor $1 + o(1)$, i.e., its log up to an additive $o(1)$.

Until Section 5.3 we will assume $d \geq 2$. (The results for $d = 1$ are simpler to derive but require some modification.)

To see how we are now positioned to obtain $-\log p$ up to additive $o(1)$, note by Sytaja's theorem and Corollary 4.4 that

$$-\log p = h(x^*) - \varepsilon x^* + \tfrac{1}{2}\log\left[2\pi(1-\xi_1)\varepsilon\tilde{x}_0\right] + o(1) \tag{5.1}$$

where $\xi_1 = 1/(2m_1+2)$ and $\tilde{x}_0$ is given explicitly by (4.2). Here x^* solves $h'(x) = \varepsilon$, and Proposition 4.5 gives an asymptotic expansion allowing its computation to an arbitrarily large power of ε once the solution x_0 to (4.1) is obtained (as discussed in Section 4.4). The expansion for x^* can then be substituted into the expansion (3.12) for $h(\cdot)$.

In this section we present for the reader's convenience some explicit small-deviations estimates in a few cases where the final result of the above program is reasonably clean. In Section 5.1 we determine lead-order asymptotics, and more, for $-\log p$. In Section 5.2 we determine lead-order asymptotics for p itself. For completeness, in Section 5.3 we handle the one-dimensional case of m-integrated Brownian motion.

5.1 Logarithmic small-deviations estimates ($d \geq 2$)

Our first two results give lead-order asymptotics for $-\log p$.

Theorem 5.1. *Let V be given as at (1.4) for an arbitrary $(m_1, \ldots, m_d)$-integrated Brownian sheet with $d \geq 2$, and assume $m := m_1 = \cdots = m_t < m_{t+1} \leq \cdots \leq m_d$. Then as $\varepsilon \to 0+$ we have*

$$-\log \mathbf{P}\left(V^2 \leq \varepsilon\right) = (1+o(1))\frac{2m+1}{2}\left[\frac{C\cdot(2m+2)^{t-2}}{(2m+1)^{t-1}}\right]^{\frac{2m+2}{2m+1}}\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2m+1}}\left(\log\frac{1}{\varepsilon}\right)^{\frac{(t-1)(2m+2)}{2m+1}},$$

where

$$C(d,t,m) \equiv C := \frac{\left[\sin\left(\frac{\pi}{2m+2}\right)\right]^{d-1-t}\prod_{j=t+1}^{d} L^*_{m_j}\left(-\frac{1}{2m+2}\right)}{(t-1)!\,[(2m+2)\pi]^{d-1}}. \tag{5.2}$$

Proof. Using Sytaja's Tauberian Theorem and Lemma 3.13 we find

$$-\log \mathbf{P}(V^2 \leq \varepsilon) = (1+o(1))\left[h(x^*) - \varepsilon\, x^*\right] = (1+o(1))2^{\xi-1}(1-\xi)c_{1,t-1}\,(x^*)^{\xi}(\log x^*)^{t-1},$$

where $\xi = 1/(2m+2)$ and $c_{1,t-1}$ is given by (3.14) and so equals C. Now use Lemma 4.3 and rearrange to obtain the desired result. □

Corollary 5.2. *If $m_1 = \cdots = m_d = m$, then as $\varepsilon \to 0+$ we have*

$$\begin{aligned}
&-\log \mathbf{P}\left(V^2 \leq \varepsilon\right) \\
&= (1+o(1))\frac{2m+1}{2}\left[\frac{(2m+2)^{-1}\csc\left(\frac{\pi}{2m+2}\right)}{(d-1)!\,[(2m+1)\pi]^{d-1}}\right]^{\frac{2m+2}{2m+1}}\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2m+1}}\left(\log\frac{1}{\varepsilon}\right)^{\frac{(d-1)(2m+2)}{2m+1}}. \quad □
\end{aligned}$$

Corollary 5.2 was previously established by Li [12]. More recently, Theorem 5.1 was obtained independently of us, and essentially simultaneously, by Karol' et. al [9] with a different form for the leading constant. Indeed, the trace sums that appear in Corollary 5.2 of [9] are simply generalized Dirichlet series values: recall our Remark 3.15.

Next we show that it is possible to obtain arbitrarily high-order corrections to the estimate for $-\log p$ in Theorem 5.1. [Note, however, that the asymptotic scale clearly is not fine enough to obtain $-\log p$ up to additive $o(1)$.]

Theorem 5.3. *The $o(1)$ expression in Theorem 5.1 has a complete asymptotic expansion of the form*

$$\sum_{r=1}^{\infty}\sum_{s=0}^{r} D_{rs}\left(\log\log\frac{1}{\varepsilon}\right)^{s}\left(\log\frac{1}{\varepsilon}\right)^{-r}.$$

The lead-order coefficient is $D_{11}=(t-1)^2(2m+2)/(2m+1)$.

Proof. We give a sketch. Using Proposition 4.5 with $j=1$, one finds that substitution of x_0 for x^* in (5.1) produces an error which is negligible relative to the scale involved in the statement of the theorem. Substitution of the expansion in Lemma 4.8 for x_0 leads to the desired result. The details are straightforward but quite tedious, and are omitted. ∎

Theorem 5.3 immediately gives us the following two examples.

Example 5.4. When $\mathbf{m}=\mathbf{0}$, Theorem 5.3 sharpens Csaki's classical result [3] for d-dimensional Brownian sheet ($d\geq 2$) to

$$\begin{aligned}-\log\mathbf{P}(V^2\leq\varepsilon) &= \frac{1}{8}\left[(d-1)!\,\pi^{d-1}\right]^{-2}\frac{1}{\varepsilon}\left(\log\frac{1}{\varepsilon}\right)^{2(d-1)} \\ &\quad\times\left[1+2(d-1)^2\,\frac{\log\log\frac{1}{\varepsilon}}{\log\frac{1}{\varepsilon}}+O\!\left(\frac{1}{\log\frac{1}{\varepsilon}}\right)\right].\end{aligned}$$

Example 5.5. For the (m,m)-integrated Brownian sheet, $m\geq 0$, we have

$$\begin{aligned}-\log\mathbf{P}(V^2\leq\varepsilon) &= \alpha\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2m+1}}\left(\log\frac{1}{\varepsilon}\right)^{\frac{2m+2}{2m+1}}+\alpha\cdot\frac{2m+2}{2m+1}\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)^{\frac{1}{2m+1}}\log\log\frac{1}{\varepsilon} \\ &\quad+O\!\left(\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)^{\frac{1}{2m+1}}\right),\end{aligned}$$

where

$$\alpha(m)\equiv\alpha:=\frac{1}{2}(2m+1)^{-\frac{1}{2m+1}}\left[\frac{\csc\left(\frac{\pi}{2m+2}\right)}{(2m+2)\pi}\right]^{\frac{2m+2}{2m+1}}.$$

5.2 Strong small-deviation estimates ($d \geq 2$)

We focus on the case of distinct m_j's. In this case ($t = 1$) the statement of Theorem 5.1 simplifies to $-\log \mathbf{P}\left(V^2 \leq \varepsilon\right) = (1 + o(1))E(\varepsilon)$, where C is given by (5.2) and $E(\varepsilon)$ is the expression

$$E(\varepsilon) = \frac{2m+1}{2}\left(\frac{C}{2m+2}\right)^{\frac{2m+2}{2m+1}}\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2m+1}}. \tag{5.3}$$

Theorem 5.6. *Let V be given as at (1.4) for an $(m_1, \dots, m_d)$-integrated Brownian sheet with $d \geq 2$, and suppose that $m := m_1 < \cdots < m_d$. Then as $\varepsilon \to 0+$ we have*

$$\mathbf{P}\left(V^2 \leq \varepsilon\right) = (1 + o(1))\left[\frac{\pi}{m+1}E(\varepsilon)\right]^{-1/2} \exp\{-E(\varepsilon) \times [1 + \Sigma(\varepsilon)]\}$$

for some finite linear combination $\Sigma(\varepsilon)$ of powers of ε, wherein each power is a nonzero nonnegative integer combination of the numbers

$$\frac{(2m_1+2)(2m_\nu+1)}{(2m_1+1)(2m_\nu+2)} - 1, \qquad \nu = 2, \dots, d.$$

Proof. This follows routinely from (5.1) and Example 4.6, recalling the notation $\eta_\nu = (2m_\nu + 1)/(2m_\nu + 2)$ for $\nu = 1, \dots, d$. We omit the details. □

Example 5.7. Even when $d = 2$, some vectors $\mathbf{m}$ require many terms in $\Sigma(\varepsilon)$ in order to obtain the small-ball probability up to a factor $1 + o(1)$. We will be content here to illustrate Theorem 5.6 by using it to state a first-order correction to Theorem 5.1 when $d = 2$ and $m_1 < m_2$. The present example thus complements Example 5.5.

For the (m_1, m_2)-integrated Brownian sheet with $m_1 < m_2$, we have

$$-\log \mathbf{P}(V^2 \leq \varepsilon) = \alpha_1\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2m_1+1}} + \alpha_2\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2m_1+1}\cdot\frac{2m_1+2}{2m_2+2}} + o\left(\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2m_1+1}\cdot\frac{2m_1+2}{2m_2+2}}\right),$$

where $\alpha_1 = (2m_1 + 1)[c_{1,0}(1)]^{\frac{2m_1+2}{2m_1+1}}$, $\alpha_2 = (2m_2 + 2)\,[c_{1,0}(1)]^{\frac{1}{2m_1+1}\cdot\frac{2m_1+2}{2m_2+2}} c_{2,0}(1)$, and $c_{1,0}(1)$ and $c_{2,0}(1)$ are given as in Lemma 3.13.

Example 5.8. Suppose, as in Example 4.7, that $m = m_1 = 0$ and that the m_j's are distinct, in which case (5.2) reduces to

$$C = (2\pi)^{-(d-1)}\prod_{j=2}^{d} L^*_{m_j}(-1/2)$$

and (5.3) simplifies further to

$$E(\varepsilon) = \frac{C^2}{8} \times \frac{1}{\varepsilon}.$$

Then, using the calculations in Example 4.7 it is easy to check that the only powers of ε appearing in $\Sigma(\varepsilon)$ of Theorem 5.6 are powers $2\eta_\nu - 1$, for $\nu = 2, \dots, d$, and $4\eta_2 - 2$; we recall $\eta_\nu := (2m_\nu + 1)/(2m_\nu + 2)$. If, in particular, $d = 2$, then only two terms are needed in $\Sigma(\varepsilon)$: see the next example.

Example 5.9. For $(0, m_2)$-integrated Brownian sheet with $m_2 > 0$, we have

$$\mathbf{P}(V^2 \le \varepsilon) = (1 + o(1)) \frac{\varepsilon^{1/2}}{c_{1,0}(1)\sqrt{\pi}} \exp\left\{ d_1 \cdot \frac{1}{\varepsilon} + d_2 \left(\frac{1}{\varepsilon}\right)^{\frac{1}{m_2+1}} + d_3 \left(\frac{1}{\varepsilon}\right)^{-\frac{m_2-1}{m_2+1}} \right\}$$

where $d_1 = [c_{1,0}(1)]^2$, $d_2 = (2m_2 + 2)[c_{1,0}(1)]^{\frac{1}{m_2+1}} c_{2,0}$, and $d_3 = [c_{1,0}(1)]^{-\frac{2m_2}{m_2+1}} [c_{2,0}(1)]^2$, and $c_{1,0}(1)$ and $c_{2,0}(1)$ are given as in Lemma 3.13.

5.3 Strong small deviation estimates when $d = 1$

The difference between the cases $d \ge 2$ and $d = 1$ can be seen by comparing (3.16) and Lemma 3.2 and noting the constant term $2 \log |\det U_m| - (m + 1) \log(2m + 2)$ in Lemma 3.2. Once this difference is noted, it is simple to obtain results for $d = 1$ like those in the preceding two subsections. In particular, the logarithmic small-deviations result in Corollary 5.2 remains true verbatim when $d = 1$. Moreover, we have the following strong small-deviations analogue, studied previously in [5], [6], [7], [13], and [14], of Theorem 5.6:

Theorem 5.10. *Let $V^2 \equiv V_m^2 := \int_0^1 X_m^2(t)\, dt$, where X_m is m-times integrated Brownian motion defined in Remark 1.1. Then as $\varepsilon \to 0+$ we have*

$$\mathbf{P}\left(V^2 \le \varepsilon\right) = (1 + o(1)) \left[|\det U_m| \cdot (2m + 2)^{(m+1)/2} \right]^{-1} \left[\frac{\pi}{m + 1} E(\varepsilon) \right]^{-1/2} \exp\{-E(\varepsilon)\}.$$

Here, with $C = \csc(\pi/(2m + 2))$, the expression $E(\varepsilon)$ is given by (5.3)*, and U_m is the Vandermonde matrix defined at* (3.3)*.*

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