On Vertex, Edge, and Vertex-Edge Random Graphs

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Abstract

We consider three classes of random graphs: edge random graphs, vertex random graphs, and vertex-edge random graphs. Edge random graphs are Erdős-Rényi random graphs, vertex random graphs are generalizations of geometric random graphs, and vertex-edge random graphs generalize both. The names of these three types of random graphs describe where the randomness in the models lies: in the edges, in the vertices, or in both. We show that vertex-edge random graphs, ostensibly the most general of the three models, can be approximated arbitrarily closely by vertex random graphs, but that the two categories are distinct.

1 Introduction

The classic random graphs are those of Erdős and Rényi [8, 9]. In their model, each edge is chosen independently of every other. The randomness inhabits the edges; vertices simply serve as placeholders to which random edges attach.

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Since the introduction of Erdős-Rényi random graphs, many other models of random graphs have been developed. For example, *random geometric graphs* are formed by randomly assigning points in a Euclidean space to vertices and then adding edges deterministically between vertices when the distance between their assigned points is below a fixed threshold; see [20] for an overview. For these random graphs, the randomness inhabits the vertices and the edges reflect relations between the randomly chosen structures assigned to them.

Finally, there is a class of random graphs in which randomness is imbued both upon the vertices and upon the edges. For example, in latent position models of social networks, we imagine each vertex as assigned to a random position in a metric "social" space. Then, given the positions, vertices whose points are near each other are more likely to be adjacent. See, for example, [2, 12, 16, 17, 19]. Such random graphs are, roughly speaking, a hybrid of Erdős-Rényi and geometric graphs.

We call these three categories, respectively, edge random, vertex random, and vertex-edge random graphs. From their formal definitions in Section 2, it follows immediately that vertex random and edge random graphs are instances of the more generous vertex-edge random graph models. But is the vertex-edge random graph category strictly more encompassing? We observe in Section 3 that a vertex-edge random graph can be approximated arbitrarily closely by a vertex random graph. Is it possible these two categories are, in fact, the same? The answer is no, and this is presented in Section 4. Our discussion closes in Section 5 with some open problems.

Nowadays, in most papers on random graphs, for each value of n a distribution is placed on the collection of n-vertex graphs and asymptotics as $n \to \infty$ are studied. We emphasize that in this paper, by contrast, the focus is on what kinds of distributions arise in certain ways for a single arbitrary but fixed value of n.

2 Random Graphs

For a positive integer n, let $[n] = \{1, 2, \ldots, n\}$ and let $\mathscr{G}_n$ denote the set of all simple graphs $G = (V, E)$ with vertex set $V = [n]$. (A simple graph is an undirected graph with no loops and no parallel edges.) We often abbreviate the edge (unordered pair) $\{i, j\}$ as ij or write $i \sim j$ and say that i and j are adjacent.

When we make use of probability spaces, we omit discussion of measurability when it is safe to do so. For example, when the sample space is finite it goes without saying that the corresponding σ-field is the total σ-field, that is, that all subsets of the sample space are taken to be measurable.

Definition 2.1 (Random graph)**.** A *random graph* is a probability space of the form $\mathbf{G} = (\mathscr{G}_n, P)$ where n is a positive integer and P is a probability measure defined on $\mathscr{G}_n$.

In actuality, we *should* define a random graph as a graph-valued random variable, that is, as a measurable mapping from a probability space into $\mathscr{G}_n$. However, the distribution of such a random object is a probability measure on $\mathscr{G}_n$ and is all that is of interest in this paper, so the abuse of terminology in Definition 2.1 serves our purposes.

Example 2.2 (Erdős-Rényi random graphs)**.** A simple random graph is the Erdős-Rényi random graph in the case $p = \frac{1}{2}$. This is the random graph $\mathbf{G} = (\mathscr{G}_n, P)$ where

$$P(G) := 2^{-\binom{n}{2}}, \quad G \in \mathscr{G}_n.$$

[Here and throughout we abbreviate $P(\{G\})$ as $P(G)$; this will cause no confusion.] More generally, an Erdős-Rényi random graph is a random graph $\mathbf{G}(n,p) = (\mathscr{G}_n, P)$ where $p \in [0,1]$ and

$$P(G) := p^{|E(G)|}(1-p)^{\binom{n}{2}-|E(G)|}, \quad G \in \mathscr{G}_n.$$

This means that the $\binom{n}{2}$ potential edges appear independently of each other, each with probability p.

This random graph model was first introduced by Gilbert [11]. Erdős and Rényi [8, 9], who started the systematic study of random graphs, actually considered the closely related model $\mathbf{G}(n,m)$ with a fixed number of edges. However, it is now common to call both models Erdős-Rényi random graphs.

Example 2.3 (Single coin-flip random graphs)**.** Another simple family of random graphs is one we call the *single coin-flip* family. Here $\mathbf{G} = (\mathscr{G}_n, P)$ where $p \in [0,1]$ and

$$P(G) := \begin{cases} p & \text{if } G = K_n, \\ 1-p & \text{if } G = \overline{K_n}, \\ 0 & \text{otherwise.} \end{cases}$$

As in the preceding example, each edge appears with probability p; but now all edges appear or none do.

In the successive subsections we specify our definitions of *edge*, *vertex*, and *vertex-edge* random graphs.

2.1 Edge random graph

In this paper, by an edge random graph (abbreviated ERG in the sequel) we simply mean a classical Erdős-Rényi random graph.

Definition 2.4 (Edge random graph)**.** An *edge random graph* is an Erdős-Rényi random graph $\mathbf{G}(n,p)$.

We shall also make use of the following generalization that allows variability in the edge-probabilities.

Definition 2.5 (Generalized edge random graph)**.** A generalized edge random graph (GERG) is a random graph for which the events that individual vertex-pairs are joined by edges are mutually independent but do not necessarily have the same probability. Thus to each pair $\{i,j\}$ of distinct vertices we associate a probability $\mathbf{p}(i,j)$ and include the edge ij with probability $\mathbf{p}(i,j)$; edge random graphs are the special case where $\mathbf{p}$ is constant.

Formally, a GERG can be described as follows. Let n be a positive integer and let $\mathbf{p} : [n] \times [n] \to [0,1]$ be a symmetric function. The *generalized edge random graph* $\mathbf{G}(n,\mathbf{p})$ is the probability space $(\mathscr{G}_n, P)$ with

$$P(G) := \prod_{\substack{i<j \\ ij \in E(G)}} \mathbf{p}(i,j) \times \prod_{\substack{i<j \\ ij \notin E(G)}} [1-\mathbf{p}(i,j)].$$

We call the graphs in these two definitions (generalized) *edge* random graphs because all of the randomness inhabits the (potential) edges. The inclusion of ERGs in GERGs is strict, as easily constructed examples show.

GERGs have appeared previously in the literature, e.g. in [1]; see also the next example and Definition 2.16 below.

Example 2.6 (Stochastic blockmodel random graphs)**.** A stochastic blockmodel random graph is a GERG in which the vertex set is partitioned into blocks $B_1, B_2, \ldots, B_b$ and the probability that vertices i and j are adjacent depends only on the blocks in which i and j reside.

A simple example is a random bipartite graph defined by partitioning the vertex set into B_1 and B_2 and taking $\mathbf{p}(i,j) = 0$ if $i,j \in B_1$ or $i,j \in B_2$, while $\mathbf{p}(i,j) = p$ (for some given p) if $i \in B_1$ and $j \in B_2$ or vice versa.

The concept of blockmodel is interesting and useful when b remains fixed and $n \to \infty$. Asymptotics of blockmodel random graphs have been considered, for example, by Söderberg [25]. (He also considers the version where the partitioning is random, constructed by independent random choices of a type in $\{1,...,b\}$ for each vertex; see Example 2.18.)

Recall, however, that in this paper we hold n fixed and note that in fact every GERG can be represented as a blockmodel by taking each block to be a singleton.

A salient feature of Example 2.6 is that vertex labels matter. Intuitively, we may expect that if all isomorphic graphs are treated "the same" by a GERG, then it is an ERG. We proceed to formalize this correct intuition, omitting the simple proof of Proposition 2.8.

Definition 2.7 (Isomorphism invariance)**.** Let $\mathbf{G} = (\mathscr{G}_n, P)$ be a random graph. We say that $\mathbf{G}$ is *isomorphism-invariant* if for all $G, H \in \mathscr{G}_n$ we have $P(G) = P(H)$ whenever G and H are isomorphic.

Proposition 2.8. *Let* $\mathbf{G}$ *be an isomorphism-invariant generalized edge random graph. Then* $\mathbf{G} = \mathbf{G}(n,p)$ *for some* n, p*. That is,* $\mathbf{G}$ *is an edge random graph.* □

2.2 Vertex random graph

The concept of a vertex random graph (abbreviated VRG) is motivated by the idea of a random intersection graph. One imagines a universe $\mathscr{S}$ of geometric objects. A random $\mathscr{S}$-graph $G \in \mathscr{G}_n$ is created by choosing n members of $\mathscr{S}$ independently at random[1], say $S_1, \ldots, S_n$, and then declaring distinct vertices i and j to be adjacent if and only if $S_i \cap S_j \neq \emptyset$. For example,

[1] Of course, some probability distribution must be associated with $\mathscr{S}$.

when $\mathscr{S}$ is the set of real intervals, one obtains a random interval graph [5, 14, 22, 23]; see Example 2.12 for more. In [10, 15, 24] one takes $\mathscr{S}$ to consist of discrete (finite) sets. Random chordal graphs can be defined by selecting random subtrees of a tree [18].

Notice that for these random graphs, all the randomness lies in the structures attached to the vertices; once these random structures have been assigned to the vertices, the edges are *determined*. In Definition 2.11 we generalize the idea of a random intersection graph to other vertex-based representations of graphs; see [29].

Definition 2.9 ($(\mathbf{x},\phi)$-graph)**.** Let n be a positive integer, $\mathscr{X}$ a set, $\mathbf{x} = (x_1,\dots,x_n)$ a function from $[n]$ into $\mathscr{X}$, and $\phi : \mathscr{X}\times\mathscr{X} \to \{0,1\}$ a symmetric function. Then the $(\mathbf{x},\phi)$*-graph*, denoted $\mathbf{G}(\mathbf{x},\phi)$, is defined to be the graph with vertex set $[n]$ such that for all $i,j \in [n]$ with $i \neq j$ we have

$$ij \in E \quad \text{if and only if} \quad \phi(x_i,x_j) = 1.$$

Of course, every graph $G = (V,E)$ with $V = [n]$ is an $(\mathbf{x},\phi)$-graph for some choice of $\mathscr{X}$, $\mathbf{x}$, and ϕ; one need only take $\mathbf{x}$ to be the identity function on $\mathscr{X} := [n]$ and define

$$\phi(i,j) := \mathbf{1}(ij \in E) = \begin{cases} 1 & \text{if } ij \in E \\ 0 & \text{otherwise.} \end{cases}$$

It is also clear that this representation of G as an $(\mathbf{x},\phi)$-graph is far from unique. The notion of $(\mathbf{x},\phi)$-graph becomes more interesting when one or more of $\mathscr{X}$, $\mathbf{x}$, and ϕ are specified.

Example 2.10 (Interval graphs)**.** Take $\mathscr{X}$ to be the set of all real intervals and define

$$\phi(J,J') := \begin{cases} 1 & \text{if } J \cap J' \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

In this case, an $(\mathbf{x},\phi)$-graph is exactly an interval graph.

Definition 2.11 (Vertex random graph)**.** To construct a vertex random graph (abbreviated VRG), we imbue $\mathscr{X}$ with a probability measure μ and sample n elements of $\mathscr{X}$ independently at random to get $\mathbf{x}$, and then we build the $(\mathbf{x},\phi)$-graph.

Formally, let n be a positive integer, $(\mathscr{X},\mu)$ a probability space, and $\phi : \mathscr{X}\times\mathscr{X} \to \{0,1\}$ a symmetric function. The *vertex random graph* $\mathbf{G}(n,\mathscr{X},\mu,\phi)$ is the random graph $(\mathscr{G}_n,P)$ with

$$P(G) := \int \mathbf{1}\{\mathbf{G}(\mathbf{x},\phi) = G\}\,\mu(d\mathbf{x}), \quad G \in \mathscr{G}_n,$$

where $\mu(d\mathbf{x})$ is shorthand for the product integrator $\mu^n(d\mathbf{x}) = \mu(dx_1)\dots\mu(dx_n)$ on $\mathscr{X}^n$.

Note that $\mathbf{G}(\cdot,\phi)$ is a graph-valued random variable defined on $\mathscr{X}^n$. The probability assigned by the vertex random graph to $G \in \mathscr{G}_n$ is simply the probability that this random variable takes the value G.

Example 2.12 (Random interval graphs)**.** Let $\mathscr{X}$ be the set of real intervals as in Example 2.10, let ϕ be as in (1), and let μ be a probability measure on $\mathscr{X}$. This yields a VRG that is a random interval graph.

Example 2.13 (Random threshold graphs)**.** Let $\mathscr{X} = [0,1]$, let μ be Lebesgue measure, and let ϕ be the indicator of a given up-set in the usual (coordinatewise) partial order on $\mathscr{X} \times \mathscr{X}$. This yields a VRG that is a random threshold graph; see [6].

Example 2.14 (Random geometric graphs)**.** Random geometric graphs are studied extensively in [20]. Such random graphs are created by choosing n i.i.d. (independent and identically distributed) points from some probability distribution on $\mathbb{R}^k$. Then, two vertices are joined by an edge exactly when they lie within a certain distance, t, of each other.

Expressed in our notation, we let $(\mathscr{X}, d)$ be a metric space equipped with a probability measure μ and let $t > 0$ (a threshold). For points $x, y \in \mathscr{X}$ define

$$\phi(x,y) := \mathbf{1}\{d(x,y) \le t\}.$$

That is, two vertices are adjacent exactly when the distance between their corresponding randomly chosen points is sufficiently small.

Because the n vertices in a vertex random graph are drawn i.i.d. from $(\mathscr{X}, \mu)$, it is easy to see that the random graph is isomorphism-invariant.

Proposition 2.15. *Every vertex random graph is isomorphism-invariant.* □

2.3 Vertex-edge random graphs

A generalization both of vertex random graphs and of edge random graphs are the *vertex-edge* random graphs (abbreviated VERGs) of Definition 2.17. First we generalize Definition 2.9 to allow edge probabilities other than 0 and 1.

Definition 2.16 (Random $(\mathbf{x}, \phi)$-graph)**.** Given a positive integer $n \ge 1$, a set $\mathscr{X}$, and a function $\phi : \mathscr{X} \times \mathscr{X} \to [0,1]$, we assign to each $i \in [n]$ a deterministically chosen object $x_i \in \mathscr{X}$. Then, for each pair $\{i, j\}$ of vertices, independently of all other pairs, the edge ij is included in the random $(\mathbf{x}, \phi)$-graph with probability $\phi(x_i, x_j)$.

Formally, let $\mathbf{x} = (x_1, \ldots, x_n)$ be a given function from $[n]$ into $\mathscr{X}$. Then the *random* $(\mathbf{x}, \phi)$-*graph*, denoted $\mathbf{G}(\mathbf{x}, \phi)$, is defined to be the random graph $(\mathscr{G}_n, P_{\mathbf{x}})$ for which the probability of $G \in \mathscr{G}_n$ is given by

$$P_{\mathbf{x}}(G) := \prod_{i<j,\ i\sim j} \phi(x_i, x_j) \times \prod_{i<j,\ i\not\sim j} [1 - \phi(x_i, x_j)].$$

Notice that $\mathbf{G}(\mathbf{x}, \phi)$ is simply the generalized edge random graph $\mathbf{G}(n, \mathbf{p})$ where $\mathbf{p}(i,j) := \phi(x_i, x_j)$ (recall Definition 2.5).

Definition 2.17 (Vertex-edge random graph)**.** Let n be a positive integer, $(\mathscr{X}, \mu)$ a probability space, and $\phi : \mathscr{X} \times \mathscr{X} \to [0,1]$ a symmetric function. In words, a vertex-edge random graph is generated like this: First a list of random elements is drawn i.i.d., with distribution μ, from $\mathscr{X}$; call the list $\mathbf{X} = (X_1, \ldots, X_n)$. Then, conditionally given $\mathbf{X}$, independently for each pair of distinct vertices i and j we include the edge ij with probability $\phi(X_i, X_j)$.

Formally, the *vertex-edge random graph* $\mathbf{G}(n, \mathscr{X}, \mu, \phi)$ is the random graph $(\mathscr{G}_n, P)$ with

$$P(G) := \int P_{\mathbf{x}}(G)\, \mu(d\mathbf{x})$$

where the integration notation is as in Definition 2.11 and $P_{\mathbf{x}}$ is the probability measure for the random $(\mathbf{x}, \phi)$-graph $\mathbf{G}(\mathbf{x}, \phi)$ of Definition 2.16.

Note that a VRG is the special case of a VERG with ϕ taking values in $\{0, 1\}$.

It can be shown [13] that every VERG can be constructed with the standard choice $\mathscr{X} = [0, 1]$ and $\mu =$ Lebesgue measure. However, other choices are often convenient in specific situations.

We note in passing that one could generalize the notions of VRG and VERG in the same way that edge random graphs (ERGs) were generalized in Definition 2.5, by allowing different functions ϕ_{ij} for different vertex pairs $\{i, j\}$. But while the notion of generalized ERG was relevant to the definition of a VERG (recall the sentence preceding Definition 2.17), we neither study nor employ generalized VRGs and VERGs in this paper.

Asymptotic properties (as $n \to \infty$) of random $(\mathbf{x}, \phi)$-graphs and VERGs have been studied by several authors: see, e.g., [3] and the references therein. VERGs are also important in the theory of *graph limits*; see for example [4, 7, 17].

Example 2.18 (Finite-type VERG)**.** In the special case when $\mathscr{X}$ is finite, $\mathscr{X} = \{1, \dots, b\}$ say, we thus randomly and independently choose a type in $\{1, \dots, b\}$ for each vertex, with a given distribution μ; we can regard this as a random partition of the vertex set into blocks $B_1, \dots, B_b$ (possibly empty, and with sizes governed by a multinomial distribution). A VERG with $\mathscr{X}$ finite can thus be regarded as a stochastic blockmodel graph with multinomial random blocks; cf. Example 2.6. Such finite-type VERGs have been considered by Söderberg [25, 26, 27, 28].

Example 2.19 (Random dot product graphs)**.** In [16, 19] random graphs are generated by the following two-step process. First, n vectors (representing n vertices) $\mathbf{v}_1, \dots, \mathbf{v}_n$ are chosen i.i.d. according to some probability distribution on $\mathbb{R}^k$. With this choice in place, distinct vertices i and j are made adjacent with probability $\mathbf{v}_i \cdot \mathbf{v}_j$. All pairs are considered (conditionally) independently. Care is taken so that the distribution on $\mathbb{R}^k$ satisfies

$$P\big(\mathbf{v}_i \cdot \mathbf{v}_j \notin [0, 1]\big) = 0.$$

Random dot product graphs are vertex-edge random graphs with $\mathscr{X} = \mathbb{R}^k$ and $\phi(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$.

As with vertex random graphs, all vertices are treated "the same" in the construction of a vertex-edge random graph.

Proposition 2.20. *Every vertex-edge random graph is isomorphism-invariant.* □

This implies that not every random graph is a VERG. A more interesting reason for this is that every VERG with $n \geq 4$ has the property that the events $\{1 \sim 2\}$ and $\{3 \sim 4\}$ are independent.

Example 2.21. The random graph $\mathbf{G}(n,m)$ considered by Erdős and Rényi [8, 9] has vertex set $[n]$ and m edges, with all such graphs having the same probability. If $n \geq 4$ and $0 < m < \binom{n}{2}$, then the events $\{1 \sim 2\}$ and $\{3 \sim 4\}$ are negatively correlated, and thus $\mathbf{G}(n,m)$ is not a VERG. [It can be shown, using Theorem 4.4, that if $0 < m < \binom{n}{2}$, then $\mathbf{G}(n,m)$ is not a VERG also when $n < 4$.]

Note that we use the notation $\mathbf{G}(n,\mathscr{X},\mu,\phi)$ for both VRGs and VERGs. This is entirely justified because ϕ takes values in in $\{0,1\}$ for VRGs and in $[0,1]$ for VERGs. If perchance the ϕ function for a VERG takes only the values 0 and 1, then the two notions coincide. Hence we have part (b) of the following proposition; part (a) is equally obvious.

Proposition 2.22.

(a) Every edge random graph is a vertex-edge random graph.

(b) Every vertex random graph is a vertex-edge random graph. □

However, not all generalized edge random graphs are vertex-edge random graphs, as simple counterexamples show.

We now ask whether the converses to the statements in Proposition 2.22 are true. The converse to Proposition 2.22(a) is false. Indeed, it is easy to find examples of VERGs that are not ERGs:

Example 2.23. We present one small class of examples of VERGs that are even VRGs, but not ERGs. Consider *random interval graphs* [5, 14, 22] $\mathbf{G}(n,\mathscr{X},\mu,\phi)$ with $n \geq 3$, $\mathscr{X}$ and ϕ as in Example 2.10, and (for $i \in [n]$) the random interval J_i corresponding to vertex i constructed as $[X_i,Y_i]$ or $[Y_i,X_i]$, whichever is nonempty, where $X_1,Y_1,\ldots,X_n,Y_n$ are i.i.d. uniform$[0,1]$ random variables. From an elementary calculation, independent of n, one finds that the events $\{1 \sim 2\}$ and $\{1 \sim 3\}$ are not independent.

The main result of this paper (Theorem 4.1; see also the stronger Theorem 4.2) is that the converse to Proposition 2.22(b) is also false. The class of vertex random graphs does not contain the class of vertex-edge random graphs; however, as shown in the next section, every vertex-edge random graph can be approximated arbitrarily closely by a vertex random graph.

An overview of the inclusions of these various categories is presented in Figure 1. The intersection VRG $\cap$ ERG is nonempty but not very interesting; by Theorems 4.2 and 4.4, the random graphs that are both ERG and VRG are just $\mathbf{G}(n,p)$ with $n \leq 3$ or $p = 0$ or $p = 1$. The other regions in Figure 1 are nonempty by Examples 2.21, 5.7, 2.23, and Theorem 4.2.

3 Approximation

The goal of this section is to show that every vertex-edge random graph can be closely approximated by a vertex random graph. Our notion of approximation is based on total variation distance. (This choice is not important. We consider a fixed n, and the space of probability measures on $\mathscr{G}_n$ is a finite-dimensional simplex, and thus compact. Hence any continuous metric on the probability measures on $\mathscr{G}_n$ is equivalent to the total variation distance, and can be used in Theorem 3.3.)

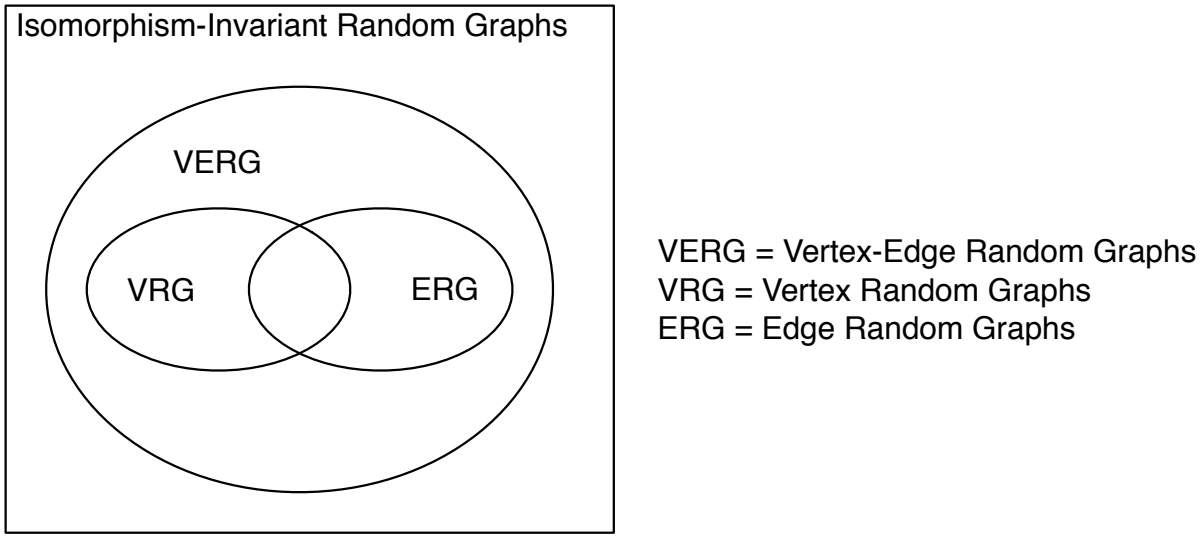

Figure 1: Venn diagram of random graph classes. The results of this paper show that all five regions in the diagram are nonempty.

Definition 3.1 (Total variation distance)**.** Let $\mathbf{G}_1 = (\mathscr{G}_n, P_1)$ and $\mathbf{G}_2 = (\mathscr{G}_n, P_2)$ be random graphs on n vertices. We define the *total variation distance* between $\mathbf{G}_1$ and $\mathbf{G}_2$ to be

$$d_{\mathrm{TV}}(\mathbf{G}_1, \mathbf{G}_2) = \frac{1}{2} \sum_{G \in \mathscr{G}_n} |P_1(G) - P_2(G)| .$$

Total variation distance can be reexpressed in terms of the maximum discrepancy of the probability of events.

Proposition 3.2. *Let $\mathbf{G}_1 = (\mathscr{G}_n, P_1)$ and $\mathbf{G}_2 = (\mathscr{G}_n, P_2)$ be random graphs on n vertices. Then*

$$d_{\mathrm{TV}}(\mathbf{G}_1, \mathbf{G}_2) = \max_{B \subseteq \mathscr{G}_n} |P_1(B) - P_2(B)| . \quad \square$$

Theorem 3.3. *Let $\mathbf{G}$ be a vertex-edge random graph and let $\varepsilon > 0$. There exists a vertex random graph $\widehat{\mathbf{G}}$ with $d_{\mathrm{TV}}(\mathbf{G}, \widehat{\mathbf{G}}) < \varepsilon$.*

We use the following simple birthday-problem subadditivity upper bound. Let M be a positive integer.

Lemma 3.4. *Let $\mathbf{A} = (A_1, A_2, \ldots, A_n)$ be a random sequence of integers with each A_i chosen independently and uniformly from $[M]$. Then*

$$P\{\mathbf{A} \text{ has a repetition}\} \leq \frac{n^2}{2M}. \quad \square$$

Proof of Theorem 3.3. Let $\mathbf{G}$ be a vertex-edge random graph on n vertices and let $\varepsilon > 0$. Let M be a large positive integer. (We postpone our discussion of just how large to take M until needed.)

The vertex-edge random graph $\mathbf{G}$ can be written $\mathbf{G} = \mathbf{G}(n, \mathscr{X}, \mu, \phi)$ for some set $\mathscr{X}$ and mapping $\phi : \mathscr{X} \times \mathscr{X} \to [0,1]$.

We construct a vertex random graph $\widehat{\mathbf{G}} = \mathbf{G}(n, \mathscr{Y}, \nu, \psi)$ as follows. Let $\mathscr{Y} := \mathscr{X} \times [0,1]^M \times [M]$; that is, $\mathscr{Y}$ is the set of ordered triples (x, f, a) where $x \in \mathscr{X}$, $f \in [0,1]^M$, and $a \in [M]$. We endow $\mathscr{Y}$ with the product measure of its factors; that is, we independently pick $x \in \mathscr{X}$ according to μ, a function $f \in [0,1]^{[M]}$ uniformly, and $a \in [M]$ uniformly. We denote this measure by ν.

We denote the components of the vector $f \in [0,1]^M$ by $f(1), \ldots, f(M)$, thus regarding f as a random function from $[M]$ into $[0,1]$. Note that for a random $f \in [0,1]^M$, the components $f(1), \ldots, f(M)$ are i.i.d. random numbers with a uniform$[0,1]$ distribution.

Next we define ψ. Let $y_1, y_2 \in \mathscr{Y}$ where $y_i = (x_i, f_i, a_i)$ (for $i = 1, 2$). Let

$$\psi(y_1, y_2) = \begin{cases} 1 & \text{if } a_1 < a_2 \text{ and } \phi(x_1, x_2) \geq f_1(a_2), \\ 1 & \text{if } a_2 < a_1 \text{ and } \phi(x_1, x_2) \geq f_2(a_1), \\ 0 & \text{otherwise.} \end{cases}$$

Note that ψ maps $\mathscr{Y} \times \mathscr{Y}$ into $\{0,1\}$ and is symmetric in its arguments. Therefore $\widehat{\mathbf{G}}$ is a vertex random graph.

We now show that $d_{\mathrm{TV}}(\mathbf{G}, \widehat{\mathbf{G}})$ can be made arbitrarily small by taking M sufficiently large.

Let $B \subseteq \mathscr{G}_n$. Recall that

$$\begin{aligned} P(B) &= \int P_{\mathbf{x}}(B)\, \mu(d\mathbf{x}), \\ \widehat{P}(B) &= \int \mathbf{1}\{\mathbf{G}(\mathbf{y}, \psi) \in B\}\, \nu(d\mathbf{y}) = \Pr\{\mathbf{G}(\mathbf{Y}, \psi) \in B\}, \end{aligned}$$

where in the last expression the n random variables comprising $\mathbf{Y} = (Y_1, \ldots, Y_n)$ are independently chosen from $\mathscr{Y}$, each according to the distribution ν.

As each Y_i is of the form (X_i, F_i, A_i) we break up the integral for $\widehat{P}(B)$ based on whether or not the a-values of the Ys are repetition free and apply Lemma 3.4:

$$\begin{aligned} \widehat{P}(B) &= \Pr\{\mathbf{G}(\mathbf{Y}, \psi) \in B \mid \mathbf{A} \text{ is repetition free}\} \Pr\{\mathbf{A} \text{ is repetition free}\} \\ &\qquad + \Pr\{\mathbf{G}(\mathbf{Y}, \psi) \in B \mid \mathbf{A} \text{ is not repetition free}\} \Pr\{\mathbf{A} \text{ is not repetition free}\} \qquad (2) \\ &= \Pr\{\mathbf{G}(\mathbf{Y}, \psi) \in B \mid \mathbf{A} \text{ is repetition free}\} + \delta \end{aligned}$$

where $|\delta| \leq n^2/(2M)$.

Now, for any repetition-free $\mathbf{a}$, the events $\{i \sim j \text{ in } \mathbf{G}(\mathbf{Y}, \psi)\}$ are conditionally independent given $\mathbf{X}$ and given $\mathbf{A} = \mathbf{a}$, with

$$\begin{aligned} \Pr\{i \sim j \text{ in } \mathbf{G}(\mathbf{Y}, \psi) \mid \mathbf{X},\ \mathbf{A} = \mathbf{a}\} &= \begin{cases} \Pr\{\phi(X_i, X_j) \geq F_i(a_j) \mid X_i, X_j\} & \text{if } a_i < a_j \\ \Pr\{\phi(X_i, X_j) \geq F_j(a_i) \mid X_i, X_j\} & \text{if } a_j < a_i \end{cases} \\ &= \phi(X_i, X_j). \end{aligned}$$

Thus, for any repetition-free $\mathbf{a}$,

$$\Pr\{\mathbf{G}(\mathbf{Y},\psi)\in B \mid \mathbf{X},\ \mathbf{A}=\mathbf{a}\}$$

equals

$$\sum_{G\in B}\left(\prod_{i<j,\ ij\in E(G)}\phi(X_i,X_j)\times\prod_{i<j,\ ij\notin E(G)}\big[1-\phi(X_i,X_j)\big]\right)=P_{\mathbf{X}}(B).$$

Removing the conditioning on $\mathbf{X}$ and $\mathbf{A}$, (2) thus implies

$$\widehat{P}(B)=P(B)+\delta,$$

and so $|P(B)-\widehat{P}(B)|\le n^2/M$ for all $B\subseteq\mathscr{G}_n$. Equivalently, $d_{\mathrm{TV}}(\mathbf{G},\widehat{\mathbf{G}})\le n^2/M$. Thus we need only choose $M>n^2/\varepsilon$. □

4 Not all vertex-edge random graphs are vertex random graphs

In Section 3 (Theorem 3.3) it was shown that every vertex-edge random graph can be approximated arbitrarily closely by a vertex random graph. This naturally raises the question of whether every vertex-edge random graph *is* a vertex random graph. We originally believed that some suitable "$M=\infty$ modification" of the proof of Theorem 3.3 would provide a positive answer, but in fact the answer is no:

Theorem 4.1. *Not all vertex-edge random graphs are vertex random graphs.*

This theorem is an immediate corollary of the following much stronger result. We say that an ERG $\mathbf{G}(n,p)$ is *nontrivial* when $p\notin\{0,1\}$.

Theorem 4.2. *If $n\ge 4$, no nontrivial Erdős-Rényi random graph is a vertex random graph. In fact, an ERG $\mathbf{G}(n,p)$ with $n\ge 4$ is represented as a vertex-edge random graph $\mathbf{G}(n,\mathscr{X},\mu,\phi)$ if and only if $\phi(x,y)=p$ for μ-almost every x and y.*

The "if" part of Theorem 4.2 is trivial (for any value of n), since $\phi(x,y)=p$ clearly gives a representation (which we shall call the *canonical* representation) of an ERG as a VERG.

We establish a lemma before proceeding to the proof of the nontrivial "only if" part of Theorem 4.2. To set up for the lemma, which relates an expected subgraph count to the spectral decomposition of a certain integral operator, consider any particular representation $\mathbf{G}(n,\mathscr{X},\mu,\phi)$ of a VERG. Let T be the integral operator with kernel ϕ on the space $L^2(\mathscr{X},\mu)$ of μ-square-integrable functions on $\mathscr{X}$:

$$(Tg)(x):=\int\phi(x,y)g(y)\,\mu(dy)=\mathrm{E}[\phi(x,X)g(X)] \tag{3}$$

where E denotes expectation and X has distribution μ. Since ϕ is bounded and μ is a finite measure, the kernel ϕ belongs to $L^2(\mu\times\mu)$. Integral operators with such kernels are Hilbert–Schmidt operators, and are thus compact operators. (See e.g. [21, Chapter VI] for the functional

analysis used here, in particular Theorems VI.23, VI.22, and VI.16.) Moreover, since ϕ is symmetric, the integral operator T is self-adjoint: For any $g,h \in L^2(\mathscr{X},\mu)$,

$$\langle Tg,h\rangle := \int Tg(x)h(x)\mu(dx) = \iint \phi(x,y)g(y)h(x)\mu(dy)\mu(dx) = \langle Th,g\rangle.$$

It follows that $L^2(\mathscr{X},\mu)$ has an orthonormal basis $(\psi_i)_{i=1,2,\dots}$ of eigenfunctions for T; these satisfy $T\psi_i = \lambda_i\psi_i$, for some (not necessarily distinct) real eigenvalues λ_i with $\lambda_i \to 0$ as $i \to \infty$. Then

$$\phi(x,y) = \sum_{i=1}^{\infty} \lambda_i \psi_i(x)\psi_i(y) \tag{4}$$

μ-a.e., with the sum converging in L^2. We may assume that λ_1 is the largest eigenvalue.

Note that in the special case $\phi(x,y) \equiv p$ giving the canonical representation of an ERG, we have $\lambda_1 = p$, $\psi_1(x) \equiv 1$, and $\lambda_i = 0$ for $i \geq 2$.

Let N_k, $3 \leq k \leq n$, be the number of rooted k-cycles in $\mathbf{G}(n,\mathscr{X},\mu,\phi)$, where a (not necessarily induced) rooted cycle is a cycle with a designated start vertex (the root) and a start direction. In the following we write $n^{\underline{k}} := n(n-1)\cdots(n-k+1)$ for the kth falling factorial power of n.

Lemma 4.3. *In a VERG, with the preceding notation, for* $3 \leq k \leq n$ *we have*

$$\mathrm{E}N_k = n^{\underline{k}} \sum_i \lambda_i^k.$$

Proof. A rooted k-cycle is given by a sequence of k distinct vertices $v_1,\dots,v_k$ with edges v_iv_{i+1} $(i = 1,\dots,k-1)$ and v_kv_1. Thus, with Tr denoting trace,

$$\begin{aligned}
\mathrm{E}N_k &= n^{\underline{k}}\mathrm{E}[\phi(X_1,X_2)\phi(X_2,X_3)\cdots\phi(X_k,X_1)] \\
&= n^{\underline{k}} \int\cdots\int_{\mathscr{X}^k} \phi(x_1,x_2)\phi(x_2,x_3)\cdots\phi(x_k,x_1)\,\mu(dx_1)\cdots\mu(dx_k) \\
&= n^{\underline{k}}\,\mathrm{Tr}\,T^k = n^{\underline{k}}\sum_i \lambda_i^k.
\end{aligned}$$

The reader unfamiliar with standard properties of Hilbert–Schmidt integral operators may wish to check directly using the expansion (4) that the integral and sum expressions in the last display are equal. □

In the special case $\phi(x,y) \equiv p$ of the canonical representation of an ERG, Lemma 4.3 reduces to

$$\mathrm{E}N_k = n^{\underline{k}}p^k, \qquad 3 \leq k \leq n, \tag{5}$$

which is otherwise clear for an ERG.

Equipped with Lemma 4.3, it is now easy to prove Theorem 4.2.

Proof of Theorem 4.2. In any VERG $\mathbf{G}(n,\mathscr{X},\mu,\phi)$, the average edge-probability ρ is given by

$$\rho := \mathrm{E}\,\phi(X_1,X_2) = \int\int \phi(x,y)\,\mu(dy)\,\mu(dx) = \langle T\mathbf{1},\mathbf{1}\rangle \leq \lambda_1,$$

where $\mathbf{1}$ is the function with constant value 1 and λ_1 is the largest eigenvalue of T; hence

$$\rho^4 \le \lambda_1^4 \le \sum_i \lambda_i^4 = \frac{\mathrm{E}N_4}{n^{\underline{4}}}, \tag{6}$$

where the equality here comes from Lemma 4.3. If the VERG is an ERG $\mathbf{G}(n,p)$, then $\rho = p$ and by combining (5) and (6) we see that $p = \rho = \lambda_1$ and $\lambda_i = 0$ for $i \ge 2$; hence, by (4), $\phi(x,y) = p\psi_1(x)\psi_1(y)$ for μ-almost every x and y, where ψ_1 is a normalized eigenfunction of T corresponding to eigenvalue $\lambda_1 = p$. But then

$$p\int \psi_1^2(x)\,\mu(dx) = p = \int\int \phi(x,y)\,\mu(dy)\,\mu(dx) = p\left[\int \psi_1(x)\,\mu(dx)\right]^2,$$

and since there is equality in the Cauchy–Schwarz inequality for ψ_1 we see that $\psi_1 = 1$ a.e. or $\psi_1 = -1$ a.e., and thus $\phi(x,y) = p$ for μ-almost every x and y. This establishes the "only if" assertion in Theorem 4.2; as already noted, the "if" assertion is trivial. □

Consider an ERG $\mathbf{G}(n,p)$. If $n \ge 4$, Theorem 4.2 shows that $\mathbf{G}(n,p)$ is never a VRG if $p \notin \{0,1\}$. Curiously, however, every $\mathbf{G}(n,p)$ with $n \le 3$ is a VRG; in fact, the following stronger result is true.

Theorem 4.4. *Every vertex-edge random graph with $n \le 3$ is a vertex random graph.*

Proof. We seek to represent the given VERG $\mathbf{G}(n,\mathscr{X},\mu,\phi)$ as a VRG $\mathbf{G}(n,\mathscr{Y},\nu,\psi)$, with ψ taking values in $\{0,1\}$. For $n = 1$ there is nothing to prove. For $n = 2$, the only random graphs of any kind are ERGs $\mathbf{G}(n,p)$; one easily checks that $\mathscr{Y} = \{0,1\}$, $\nu(1) = \sqrt{p} = 1 - \nu(0)$, and $\psi(y_1,y_2) = \mathbf{1}(y_1 = y_2 = 1)$ represents $\mathbf{G}(n,p)$ as a VRG.

Suppose now that $n = 3$. The given VERG can be described as choosing X_1, X_2, X_3 i.i.d. from μ and, independently, three independent uniform$[0,1)$ random variables U_{12}, U_{13}, U_{23}, and then including each edge ij if and only if the corresponding U_{ij} satisfies $U_{ij} \le \phi(X_i, X_j)$. According to Lemma 4.5 to follow, we can obtain such U_{ij}'s by choosing independent uniform$[0,1)$ random variables U_1, U_2, U_3 and setting $U_{ij} := U_i \oplus U_j$, where $\oplus$ denotes addition modulo 1. It follows that the given VERG is also the VRG $\mathbf{G}(3,\mathscr{Y},\nu,\psi)$, where $\mathscr{Y} := \mathscr{X} \times [0,1)$, ν is the product of μ and the uniform$[0,1)$ distribution, and, with $y_i = (x_i, u_i)$,

$$\psi(y_1,y_2) = \mathbf{1}(u_1 \oplus u_2 \le \phi(x_1,x_2)). \tag{7}$$

□

Lemma 4.5. *If U_1, U_2, U_3 are independent uniform$[0,1)$ random variables, then so are $U_1 \oplus U_2$, $U_1 \oplus U_3$, $U_2 \oplus U_3$, where $\oplus$ denotes addition modulo* 1.

Proof. The following proof seems to be appreciably simpler than a change-of-variables proof. For other proofs, see Remark 4.7 below. Let $J := \{0,\dots,k-1\}$. First check that, for k odd, the mapping

$$(z_1,z_2,z_3) \mapsto (z_1+z_2, z_1+z_3, z_2+z_3),$$

from $J \times J \times J$ into $J \times J \times J$, with addition here modulo k, is bijective. Equivalently, if U_1, U_2, U_3 are iid uniform$[0,1)$, then the joint distribution of

$$\begin{aligned} Z_{12}(k) &:= \lfloor kU_1 \rfloor + \lfloor kU_2 \rfloor, \\ Z_{13}(k) &:= \lfloor kU_1 \rfloor + \lfloor kU_3 \rfloor, \\ Z_{23}(k) &:= \lfloor kU_2 \rfloor + \lfloor kU_3 \rfloor \end{aligned}$$

is the same as that of

$$\lfloor kU_1 \rfloor, \lfloor kU_2 \rfloor, \lfloor kU_3 \rfloor.$$

Dividing by k and letting $k \to \infty$ through odd values of k gives the desired result. □

Remark 4.6. Theorem 4.4 has an extension to hypergraphs. Define a VERHG (vertex-edge random hypergraph) on the vertices $\{1, \ldots, n\}$ in similar fashion to VERGs, except that now each of the n possible hyperedges joins a subset of vertices of size $n-1$. Define a VRHG (vertex random hypergraph) similarly. Then VERHGs and VRHGs are the same, for each fixed n. The key to the proof is the observation (extending the case $n = 3$ of Lemma 4.5) that if $U_1, U_2, \ldots U_n$ are i.i.d. uniform$[0,1)$, then the same is true (modulo 1) of $S - U_1, S - U_2, \ldots, S - U_n$, where $S := U_1 + U_2 + \cdots + U_n$. The observation can be established as in the proof of Lemma 4.5, now by doing integer arithmetic modulo k, where $n-1$ and k are relatively prime, and passing to the limit as $k \to \infty$ through such values. [For example, consider $k = m(n-1)+1$ and let $m \to \infty$.]

Remark 4.7. Consider again Lemma 4.5 and its extension in Remark 4.6. Let $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ denote the circle. We have shown that the mapping $\mathbf{u} \mapsto A\mathbf{u}$ preserves the uniform distribution on $\mathbf{T}^n$, where for example in the case $n = 3$ the matrix A is given by

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

More generally, the mapping $\mathbf{u} \mapsto A\mathbf{u}$ preserves the uniform distribution on $\mathbf{T}^n$ whenever A is a nonsingular $n \times n$ matrix of integers. Indeed, then $A : \mathbf{R}^n \to \mathbf{R}^n$ is surjective, so $A : \mathbf{T}^n \to \mathbf{T}^n$ is surjective; and any homomorphism of a compact group (here $\mathbf{T}^n$) onto a compact group (here also $\mathbf{T}^n$) preserves the uniform distribution, i.e., the (normalized) Haar measure. (This follows, e.g., because the image measure is translation invariant.) This preservation can also be seen by Fourier analysis: For the i.i.d. uniform vector $\mathbf{U} = (U_1, \ldots, U_n)$ and any integer vector $\mathbf{k} = (k_1, \ldots, k_n) \neq \mathbf{0}$,

$$\mathrm{E}\exp(2\pi i \mathbf{k} \cdot A\mathbf{U}) = \mathrm{E}\exp(2\pi i A^T \mathbf{k} \cdot \mathbf{U}) = 0$$

because $A^T \mathbf{k} \neq 0$.

Remark 4.8. In this remark we (a) give a spectral characterization of all representations of a three-vertex ERG $\mathbf{G}(3,p)$ as a VERG $\mathbf{G}(3, \mathscr{X}, \mu, \phi)$ and (b) briefly discuss the spectral decomposition of the "addition modulo 1" kernel specified by (7) when $\phi(x_1, x_2) \equiv p$.

(a) Consider a VERG $\mathbf{G}(3, \mathscr{X}, \mu, \phi)$ representing an ERG $\mathbf{G}(3,p)$. It can be shown easily that p is an eigenvalue (say, $\lambda_1 = p$) with constant eigenfunction $\mathbf{1}$. [This can be done by using

the Cauchy–Schwarz inequality to prove that for any VERG with $n \geq 3$ we have the positive dependence

$$\Pr\{1 \sim 2 \text{ and } 1 \sim 3\} \geq (\Pr\{1 \sim 2\})^2, \tag{8}$$

with equality if and only if the constant function $\mathbf{1}$ is an eigenfunction of T with eigenvalue $\Pr\{1 \sim 2\}$; moreover, we have equality in (8) for an ERG. Cf. the proof of Theorem 4.2, where a similar argument is used for $n \geq 4$.] One then readily computes that the expected number of rooted cycles on three vertices is $6\sum \lambda_i^3 = 6p^3$ [this is Lemma 4.3 and (5), recalling that $n = 3$] and similarly that the expected number of rooted edges is $6\lambda_1 = 6p$ and the expected number of rooted paths on three vertices is $6\lambda_1^2 = 6p^2$. So

$$\sum_{i \geq 2} \lambda_i^3 = 0. \tag{9}$$

Conversely, suppose that a VERG $\mathbf{G}(3, \mathscr{X}, \mu, \phi)$ has eigenvalue $\lambda_1 = p$ with corresponding eigenfunction 1, and that (9) holds. Then the expected counts of rooted edges, rooted 3-paths, and rooted 3-cycles all agree with those for an ERG $\mathbf{G}(3, p)$. Since these three expected counts are easily seen to characterize any isomorphism-invariant random graph model on three vertices, the VERG represents the ERG $\mathbf{G}(3, p)$.

Summarizing, we see that a VERG $\mathbf{G}(3, \mathscr{X}, \mu, \phi)$ represents $\mathbf{G}(3, p)$ if and only if $\lambda_1 = p$ with eigenfunction 1 and (9) holds.

In particular, one can take μ to be the uniform distribution on $\mathscr{X} = [0, 1)$ and

$$\phi(x_1, x_2) = g(x_1 \oplus x_2), \quad x_1, x_2 \in [0, 1),$$

for any $g \geq 0$ satisfying $\int g(x)\,dx = p$. It follows by Lemma 4.5 that then $\mathbf{G}(3, \mathscr{X}, \mu, \phi) = \mathbf{G}(3, p)$. Alternatively, we can verify (9) by Fourier analysis as follows.

Let $e_k(x) = e^{2\pi ikx}$. Then

$$(Te_k)(x) = \int_0^1 g(x \oplus y) e_k(y)\,dy = \int_0^1 g(y) e_k(y - x)\,dy = \hat{g}(-k) e_{-k}(x), \;\; k \in \mathbb{Z}.$$

For $k = 0$, this says again that $e_0 = 1$ is an eigenfunction with eigenvalue $\hat{g}(0) = p$. For $k \geq 1$, since $\hat{g}(k) = \overline{\hat{g}(-k)}$, it follows that if $\omega_k := \hat{g}(k)/|\hat{g}(k)|$ (with $\omega_k := 1$ if this expression would give $0/0$), then $\omega_k e_k \pm e_{-k}$ are eigenfunctions with eigenvalues $\pm|\hat{g}(k)|$. Since $\{e_k\}$ is an orthonormal basis, these eigenfunctions span a dense subspace of $L^2[0, 1)$, so we have found all eigenvalues, viz. $\lambda_1 = p$ and $\pm|\hat{g}(k)|$, $k = 1, 2, \ldots$, and (9) follows.

(b) The choice $g(x) = \mathbf{1}(x \leq p)$ in (a) was used at (7) (when the VERG in question there is an ERG). In this case,

$$\hat{g}(k) = \int_0^p e^{-2\pi ikx}\,dx = \frac{1 - e^{-2\pi ikp}}{2\pi ik}$$

and the multiset of eigenvalues can be listed as (changing the numbering) $\{\lambda_j : j \in \mathbb{Z}\}$, where

$$\lambda_j := \begin{cases} \frac{|1 - e^{-2\pi ijp}|}{2\pi j} = \frac{|\sin(\pi jp)|}{\pi j}, & j \neq 0, \\ p, & j = 0. \end{cases}$$

5 Open problems

Call a VERG $\mathbf{G}(n, \mathscr{X}, \mu, \phi)$ *binary* if $\Pr\{\phi(X_1, X_2) \in \{0,1\}\} = 1$ where X_1 and X_2 are independent draws from μ. Since μ-null sets do not matter, this amounts to saying that ϕ gives a representation of the random graph as a VRG. We will make use of the observation that

$$\phi \text{ is binary if and only if } \mathrm{E}[\phi(X_1, X_2)(1 - \phi(X_1, X_2))] = 0. \tag{10}$$

In Theorem 4.4 we have seen that every VERG with $n \leq 3$ is a VRG, but what is the situation when $n \geq 4$?

Open Problem 5.1. *Is there any VRG with $n \geq 4$ that also has a non-binary VERG representation?*

Theorem 4.2 rules out constant-valued non-binary VERG representations ϕ, and the main goal now is to see what other VERGs we can rule out as VRGs. In the following proposition, X_1 and X_2 (respectively, Y_1 and Y_2) are independent draws from μ (respectively, ν).

Proposition 5.2. *If a VRG $\mathbf{G}(n, \mathscr{Y}, \nu, \psi)$ has a representation as a VERG $\mathbf{G}(n, \mathscr{X}, \mu, \phi)$, then ϕ is binary if and only if $\mathrm{E}\,\psi^2(Y_1, Y_2) = \mathrm{E}\,\phi^2(X_1, X_2)$.*

Proof. Because $\mathbf{G}(n, \mathscr{Y}, \nu, \psi)$ and $\mathbf{G}(n, \mathscr{X}, \mu, \phi)$ represent the same random graph, we have

$$\mathrm{E}\,\psi(Y_1, Y_2) = \Pr\{1 \sim 2\} = \mathrm{E}\,\phi(X_1, X_2).$$

Thus, by (10), ϕ is binary if and only if

$$0 = \mathrm{E}[\psi(Y_1, Y_2)(1 - \psi(Y_1, Y_2))] = \mathrm{E}\,\psi(Y_1, Y_2) - \mathrm{E}\,\psi^2(Y_1, Y_2)$$

agrees with

$$\mathrm{E}[\phi(X_1, X_2)(1 - \phi(X_1, X_2))] = \mathrm{E}\,\psi(Y_1, Y_2) - \mathrm{E}\,\phi^2(X_1, X_2),$$

i.e., if and only if $\mathrm{E}\,\psi^2(Y_1, Y_2) = \mathrm{E}\,\phi^2(X_1, X_2)$. □

The expression $\mathrm{E}\,\phi^2(X_1, X_2)$ is the squared Hilbert–Schmidt norm of the operator T defined at (3) and equals the sum $\sum_i \lambda_i^2$ of squared eigenvalues. So the proposition has the following corollary.

Corollary 5.3. *If a VRG $\mathbf{G}(n, \mathscr{Y}, \nu, \psi)$ has a representation as a VERG $\mathbf{G}(n, \mathscr{X}, \mu, \phi)$, and if the respective multisets of nonzero squared eigenvalues of the integral operators associated with ψ and ϕ are the same, then ϕ is binary.* □

Open Problem 5.4. *Is there any VERG with $n \geq 4$ having two representations with distinct multisets of nonzero squared eigenvalues?*

By Corollary 5.3, a positive answer to Open Problem 5.1 would imply a positive answer to Open Problem 5.4.

Our next result, Proposition 5.5, goes a step beyond Theorem 4.2. We say that ϕ is of rank r when the corresponding integral operator (3) has exactly r nonzero eigenvalues (counting multiplicities); thus the sum in (4) has r nonzero terms. For ϕ to be of rank at most 1 it is equivalent that there exists $0 \le g \le 1$ (μ-a.e.) such that (for μ-almost every x_1 and x_2)

$$\phi(x_1, x_2) = g(x_1)g(x_2). \tag{11}$$

Proposition 5.5. *For $n \ge 6$, no non-binary VERG $\mathbf{G}(n, \mathscr{X}, \mu, \phi)$ with ϕ of rank at most 1 is a VRG.*

Proof. Of course ϕ cannot be both non-binary and of rank 0. By Corollary 5.3 it suffices to show, as we will, that

($*$) *any* VERG-representation $\mathbf{G}(n, \mathscr{Y}, \nu, \psi)$ of a VERG $\mathbf{G}(n, \mathscr{X}, \mu, \phi)$ with $n \ge 6$ and ϕ of rank 1 must have the same single nonzero eigenvalue (without multiplicity).

Indeed, to prove ($*$), express ϕ as at (11) and let $\lambda_1, \lambda_2, \dots$ denote the eigenvalues corresponding to ψ. By equating the two expressions for $\mathrm{E}N_k$ obtained by applying Lemma 4.3 both to $\mathbf{G}(n, \mathscr{X}, \mu, \phi)$ and to $\mathbf{G}(n, \mathscr{Y}, \nu, \psi)$, we find, with

$$c := \left[\mathrm{E}\,\phi^2(X_1, X_2)\right]^{1/2} > 0$$

for shorthand, that

$$\sum_i \lambda_i^k = c^k, \qquad 3 \le k \le n. \tag{12}$$

Applying (12) with $k = 4$ and $k = 6$, it follows from Lemma 5.6 to follow (with $b_i := \lambda_i^4$ and $t = 3/2$) that ψ is of rank 1, with nonzero eigenvalue c. □

The following lemma, used in the proof of Proposition 5.5, is quite elementary and included for the reader's convenience.

Lemma 5.6. *If $b_1, b_2, \dots$ form a finite or infinite sequence of nonnegative numbers and $t \in (1, \infty)$, then*

$$\Big(\sum_i b_i\Big)^t \ge \sum_i b_i^t,$$

with strict inequality if more than one b_i is positive and the right-hand sum is finite.

Proof. The lemma follows readily in general from the special case of two bs, b_1 and b_2. Since the case that $b_1 = 0$ is trivial, we may suppose that $b_1 > 0$. Fix such a b_1, and consider the function

$$f(b_2) := (b_1 + b_2)^t - b_1^t - b_2^t$$

of $b_2 \ge 0$. Then $f(0) = 0$ and

$$f'(b_2) = t[(b_1 + b_2)^{t-1} - b_2^{t-1}] > 0.$$

The result follows. □

Example 5.7. Consider a VERG $\mathbf{G}(n,\mathscr{X},\mu,\phi)$ with $n \geq 6$ and $\phi(x_1,x_2) = g(x_1)g(x_2)$ for some function $g : \mathscr{X} \to (0,1)$ that is not μ-a.e. constant. By Proposition 5.5, this is not a VRG. Furthermore, an elementary calculation shows that $\Pr(1 \sim 2) = \Pr(1 \sim 3) = \left(\int g(x)\mu(dx)\right)^2$ and

$$\begin{aligned}\Pr(1 \sim 2 \text{ and } 1 \sim 3) &= \iiint g(x_1)^2 g(x_2)g(x_3)\mu(dx_1)\mu(dx_2)\mu(dx_3) \\ &= \int g(x)^2\mu(dx)\Big(\int g(x)\mu(dx)\Big)^2 > \Pr(1 \sim 2)\Pr(1 \sim 3);\end{aligned}$$

thus the events $\{1 \sim 2\}$ and $\{1 \sim 3\}$ are dependent and the VERG is not an ERG. Hence, a VERG of this type is neither a VRG nor an ERG.

With the hypothesis of Proposition 5.5 strengthened to $n \geq 8$, we can generalize that proposition substantially as follows.

Proposition 5.8. *For $1 \leq r < \infty$ and $n \geq 4(r+1)$, no non-binary VERG $\mathbf{G}(n,\mathscr{X},\mu,\phi)$ with ϕ of rank at most r is a VRG.*

It suffices to consider ϕ of rank r exactly. The strategy for proving Proposition 5.8 is essentially the same as for Proposition 5.5: Under the stated conditions on n and r, we will show that *any* VERG-representation $\mathbf{G}(n,\mathscr{Y},\nu,\psi)$ of a VERG $\mathbf{G}(n,\mathscr{X},\mu,\phi)$ with ϕ of rank r must have the same finite multiset of nonzero squared eigenvalues; application of Corollary 5.3 then completes the proof. The following two standard symmetric-function lemmas are the basic tools we need; for completeness, we include their proofs.

Lemma 5.9. *Consider two summable sequences $a_1,a_2,\dots$ and $b_1,b_2,\dots$ of strictly positive numbers; each sequence may have either finite or infinite length. For $1 \leq k < \infty$, define the elementary symmetric functions*

$$s_k := \sum_{i_1<i_2<\cdots<i_k} a_{i_1}a_{i_2}\dots a_{i_k}, \qquad t_k := \sum_{j_1<j_2<\cdots<j_k} b_{j_1}b_{j_2}\dots b_{j_k}. \tag{13}$$

For any $1 \leq K < \infty$, if $\sum_i a_i^k = \sum_j b_j^k$ for $k = 1,2,\dots,K$, then (a) *$s_k = t_k$ for $k = 1,2,\dots,K$ and* (b) *the sequence* **a** *has length $\geq K$ if and only if the sequence* **b** *does.*

Proof. Clearly all the sums $\sum a_i^k$, $\sum b_j^k$, s_k, t_k are finite, for any $k \geq 1$. Using inclusion–exclusion, each s_k can be expressed as a finite linear combination of finite products of $\sum_i a_i^1, \sum_i a_i^2, \dots \sum_i a_i^k$. (This is true when all indices i for a_i are restricted to a finite range, and so also without such a restriction, by passage to a limit.) Each t_k can be expressed in just the same way, with the sums $\sum_j b_j^m$ substituting for the respective sums $\sum_i a_i^m$. The assertion (a) then follows; and since the sequence **a** has length $\geq K$ if and only if $s_K > 0$, and similarly for **b**, assertion (b) also follows. □

Lemma 5.10. *Let $1 \leq K < \infty$, and let $a_1,\dots,a_K$ and $b_1,\dots,b_K$ be numbers. If the sums s_k and t_k defined at* (13) *satisfy $s_k = t_k$ for $k = 1,\dots,K$, then the multisets $\{a_1,\dots,a_K\}$ and $\{b_1,\dots,b_K\}$ are equal.*

Proof. We remark that the numbers a_k and b_k need not be positive, and may even be complex. The result is obvious from the identity

$$(z-a_1)\cdots(z-a_K) = z^K - s_1 z^{K-1} + s_2 z^{K-2} + \cdots + (-1)^K s_K. \qquad \square$$

Proof of Proposition 5.8. Consider a VERG $\mathbf{G}(n,\mathscr{X},\mu,\phi)$ such that ϕ has rank r, and let $M = \{\lambda_1^2, \lambda_2^2, \dots, \lambda_r^2\}$ be its multiset of nonzero squared eigenvalues. Suppose that the same random graph can also be represented as the VERG $\mathbf{G}(n,\mathscr{Y},\nu,\psi)$, and let the finite or infinite multiset $\widetilde{M} := \{\tilde{\lambda}_1^2, \tilde{\lambda}_2^2, \dots\}$ be the multiset of nonzero squared eigenvalues for ψ. As discussed immediately following the statement of the proposition, it suffices to show that the multisets M and $\widetilde{M}$ are equal.

Let $a_i := \lambda_i^4$ and $b_j := \tilde{\lambda}_j^4$. Applying Lemma 4.3 with $k = 4, 8, \dots, 4(r+1)$, we see that the hypotheses of Lemma 5.9 are satisfied for $K = r$ and for $K = r+1$. Therefore, $\widetilde{M}$ has size r and the sums (13) satisfy $s_k = t_k$ for $k = 1, 2, \dots, r$. By Lemma 5.10, the two multisets are equal. $\square$

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