Combinatorial/probabilistic analysis of a class of search-tree functionals

Jim Fill

jimfill@jhu.edu

http://www.mts.jhu.edu/~fill/

Mathematical Sciences
The Johns Hopkins University
This is joint work (in progress) with

- Philippe Flajolet, INRIA
- Nevin Kapur, JHU
Goal

For additive functionals on random $m$-ary search trees:

- derive asymptotic approximations to distributions
Goal

For additive functionals on random $m$-ary search trees:

- derive asymptotic approximations to distributions
- obtain rates of convergence (still to do)
For additive functionals on random $m$-ary search trees:

- derive asymptotic approximations to distributions
- obtain rates of convergence (still to do)
- study relation between input “toll sequence” and functional
Goal

For additive functionals on random $m$-ary search trees:

- derive asymptotic approximations to distributions
- obtain rates of convergence (still to do)
- study relation between input “toll sequence” and functional
- get *complete* asymptotic expansions of moments

Tools:
Goal

For additive functionals on random $m$-ary search trees:

- derive asymptotic approximations to distributions
- obtain rates of convergence (still to do)
- study relation between input “toll sequence” and functional
- get *complete* asymptotic expansions of moments

Tools:

- combinatorial/probabilistic “transfer results” linking asymptotics of “toll sequence” to that of functional
Goal

For additive functionals on random $m$-ary search trees:

- derive asymptotic approximations to distributions
- obtain rates of convergence (still to do)
- study relation between input “toll sequence” and functional
- get *complete* asymptotic expansions of moments

Tools:

- combinatorial/probabilistic “transfer results” linking asymptotics of “toll sequence” to that of functional
- singularity analysis (based on complex analysis)
*m*-ary search trees

- fundamental data structure in computer science
- generalizes the concept of a binary search tree

```
1 4 5
2 3

8 9
6

11

13 14 15
7 10 12
```
Each sequence of \( n \) distinct keys can be associated with an \( m \)-ary search tree got by inserting successive elements of the sequence into an initially empty tree.
each sequence of $n$ distinct keys can be associated with an $m$-ary search tree got by inserting successive elements of the sequence into an initially empty tree.
Probability models

- uniform model – each tree equally likely (future work)
Probability models

- Uniform model – each tree equally likely (future work)
- Random permutation model – each permutation on $[n] := \{1, \ldots, n\}$ equally likely (this talk)
Probability models

- **uniform model** – each tree equally likely (future work)
- **random permutation model** – each permutation on $[n] := \{1, \ldots, n\}$ equally likely (this talk)

$Q$: probability mass function of the distribution induced by associating an $m$-ary search tree with each of the $n!$ permutations on $[n]$
Probability models

- uniform model – each tree equally likely (future work)
- random permutation model – each permutation on \([n] := \{1, \ldots, n\}\) equally likely (this talk)

\(Q\): probability mass function of the distribution induced by associating an \(m\)-ary search tree with each of the \(n!\) permutations on \([n]\)

\(Q\) is a crude measure of the “shape” of a tree – balanced tree \(\Rightarrow\) large \(Q\)
Probability models

- uniform model – each tree equally likely (future work)
- random permutation model – each permutation on 
  \([n] := \{1, \ldots, n\}\) equally likely (this talk)

\(Q\): probability mass function of the distribution induced by associating an \(m\)-ary search tree with each of the \(n!\) permutations on \([n]\)

\(Q\) is a crude measure of the “shape” of a tree – balanced tree \(\Rightarrow\) large \(Q\)

(Dobrow & Fill, 1995) \(Q(T) = \frac{1}{\prod_{x \text{ full}} \binom{|T(x)|}{m-1}}\)
Probability models

- uniform model – each tree equally likely (future work)
- random permutation model – each permutation on \([n] := \{1, \ldots, n\}\) equally likely (this talk)
- \(Q\): probability mass function of the distribution induced by associating an \(m\)-ary search tree with each of the \(n!\) permutations on \([n]\)
- \(Q\) is a crude measure of the “shape” of a tree – balanced tree \(\Rightarrow\) large \(Q\)
- (Dobrow & Fill, 1995) \(Q(T) = 1/\prod_{x \text{ full}} \binom{|T(x)|}{m-1}\)
- shape functional: \(\Lambda(T) := -\ln Q(T)\)
| $|T| = 15$ |

| “full” nodes contain $m - 1$ keys |
Additive-type tree functionals

$$f(T) = \sum_{i=1}^{m} f(T_i) + c|T|, \quad |T| \geq m - 1$$

$$(c_n)_{n \geq m-1} : \text{“toll sequence”}$$

think of toll sequence as input and additive functional as output
Additive-type tree functionals

\[ f(T) = \sum_{i=1}^{m} f(T_i) + c_{|T|}, \quad |T| \geq m - 1 \]

\((c_n)_{n \geq m-1} : \) “toll sequence”

- think of toll sequence as input and additive functional as output

**Examples:**

1. \( c_n := 1 \): space requirement (Mahmoud & Pittel, 1989; Chern & Hwang, 2001)
2. \( c_n := n - (m - 1) \): internal path length (Mahmoud, 1992)  
   [analysis of \( m \)-ary Quicksort!]
3. \( c_n := \ln \left( \frac{n}{m-1} \right) \): shape functional
1. limit laws

center the functional using lead order term of the mean
1. limit laws

- center the functional using lead order term of the mean
- observe that all moments of the centered functional satisfy a recurrence relation of the same form
1. limit laws

- center the functional using lead order term of the mean
- observe that all moments of the centered functional satisfy a recurrence relation of the same form
- use a general (combinatorial) transfer theorem to get asymptotics for all moments of the centered functional
1. limit laws

- center the functional using lead order term of the mean
- observe that all moments of the centered functional satisfy a recurrence relation of the same form
- use a general (combinatorial) transfer theorem to get asymptotics for all moments of the centered functional
- conclude convergence in distribution using method of moments (when possible)
1. *limit laws*

- center the functional using lead order term of the mean
- observe that *all* moments of the centered functional satisfy a recurrence relation of the same form
- use a general (combinatorial) transfer theorem to get asymptotics for all moments of the centered functional
- conclude convergence in distribution using method of moments (when possible)
- alternative method: *contraction method* (not today)
Details: distn. of subtree size

recall

\[ f(T) = \sum_{i=1}^{m} f(T_i) + c|T| \]
Details: distn. of subtree size

- recall

\[ f(T) = \sum_{i=1}^{m} f(T_i) + c_{|T|} \]

- for \(|T| = n\), and all \(i, j\),

\[ P(|T| = j) = \binom{n - j - 1}{m - 2} / \binom{n}{m - 1} \]
Details: distn. of subtree size

- recall

\[ f(T) = \sum_{i=1}^{m} f(T_i) + c_{|T|} \]

- for \(|T| = n\), and all \(i, j\),

\[ P(|T| = j) = \binom{n-j-1}{m-2} / \binom{n}{m-1} \]

- so, for \(n \geq m - 1\), \(\mu_n := Ef(T)\) satisfies

\[ \mu_n = \frac{m}{\binom{n}{m-1}} \sum_{j} \binom{n-j-1}{m-2} \mu_j + c_n \]
the sequence \((a_n)\) of moments of each specified order satisfy a recurrence relation of the form

\[
a_n = \frac{m}{n} \sum_j \binom{n-j-1}{m-2} a_j + b_n
\]
the sequence \((a_n)\) of moments of each specified order satisfy a recurrence relation of the form

\[
a_n = \frac{m}{\binom{n}{m-1}} \sum_j \left(\binom{n-j-1}{m-2}\right) a_j + b_n
\]

equivalent to linear differential equation with characteristic polynomial

\[
\psi(\lambda) := \lambda(\lambda + 1) \cdots (\lambda + m - 2) - m!
\]

with roots \(2 = \lambda_1, \ldots, \lambda_{m-1}\)
Details: explicit solution!

for simplicity, consider case \( b_1 = \cdots = b_{m-2} = 0 \)
Details: explicit solution!

- for simplicity, consider case \( b_1 = \cdots = b_{m-2} = 0 \)
- solution easiest to state using generating functions:

\[
A(z) = B(z) \\
+ m! \sum_{j=1}^{m-1} \frac{(1 - z)^{-\lambda_j}}{\psi'(\lambda_j)} \int_{\zeta=0}^{z} B(\zeta)(1 - \zeta)^{\lambda_j - 1} d\zeta
\]
for simplicity, consider case $b_1 = \cdots = b_{m-2} = 0$

solution easiest to state using generating functions:

$$A(z) = B(z) + m! \sum_{j=1}^{m-1} \frac{(1 - z)^{-\lambda_j}}{\psi' (\lambda_j)} \int_0^z B(\zeta) (1 - \zeta)^{\lambda_j - 1} d\zeta$$

for general initial conditions, get extra term of form

$$\sum_{j=1}^{m-1} r_j (1 - z)^{-\lambda_j}$$
Transfer theorem examples

Transfer for small input:

\[ b_n = o(n) \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{b_j}{(n + 1)(n + 2)} \text{ converges} \]

transfers to

\[ a_n = \frac{K_1}{H_m - 1} n + o(n), \quad \text{where} \quad K_1 := \sum_{j=0}^{\infty} \frac{b_j}{(j + 1)(j + 2)}. \]
Transfer theorem examples

- **transfer for small input:**

  \[ b_n = o(n) \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{b_j}{(n + 1)(n + 2)} \text{ converges} \]

  transfers to

  \[ a_n = \frac{K_1}{H_m - 1} n + o(n), \quad \text{where} \quad K_1 := \sum_{j=0}^{\infty} \frac{b_j}{(j + 1)(j + 2)}. \]

- **one refinement:** if \( m \leq 26 \) and \( b_n = o(\sqrt{n}) \), then

  \[ a_n = \frac{K_1}{H_m - 1} n + o(\sqrt{n}) \]
General transfer theorem

links the asymptotic behavior of the remainder sequence \((b_n)\) (input) to \((a_n)\) (output)
General transfer theorem

links the asymptotic behavior of the remainder sequence \((b_n)\) (input) to \((a_n)\) (output)

<table>
<thead>
<tr>
<th>Input ((b_n))</th>
<th>Output ((a_n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b_n = o(n)) etc.</td>
<td>(a_n \sim \frac{K_1}{H_{m-1}} n)</td>
</tr>
<tr>
<td>(b_n \sim Kn)</td>
<td>(a_n \sim \frac{K}{H_{m-1}} n \log n)</td>
</tr>
<tr>
<td>(b_n \sim Kn^v, \ \mathfrak{R}(v) &gt; 1)</td>
<td>(a_n \sim \frac{K}{1 - \frac{m! \Gamma(v+1)}{\Gamma(v+m)}} n^v)</td>
</tr>
</tbody>
</table>
General transfer theorem

- links the asymptotic behavior of the remainder sequence \((b_n)\) (input) to \((a_n)\) (output)

<table>
<thead>
<tr>
<th>Input ((b_n))</th>
<th>Output ((a_n))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(b_n = o(n)) etc.</td>
<td>(a_n \sim \frac{K_1}{H_{m-1}} n)</td>
</tr>
<tr>
<td>(b_n \sim K n)</td>
<td>(a_n \sim \frac{K}{H_{m-1}} n \log n)</td>
</tr>
<tr>
<td>(b_n \sim K n^v, \quad \Re(v) &gt; 1)</td>
<td>(a_n \sim \frac{K}{1 - \frac{m! \Gamma(v+1)}{\Gamma(v+m)}} n^v)</td>
</tr>
</tbody>
</table>

- refined transfers available; require additional conditions
“small” toll sequence: \( c_n = O(n^{1/2}L(n)) \):

Distributional asymptotics & phase changes
"small" toll sequence: $c_n = O(n^{1/2}L(n))$:

- asymptotic normality if $m \leq 26$
"small" toll sequence: $c_n = O(n^{1/2}L(n))$:
- asymptotic normality if $m \leq 26$
- periodicity (typically) if $m \geq 27$
Distributional asymptotics & phase changes

- "small" toll sequence: $c_n = O(n^{1/2}L(n))$:
  - asymptotic normality if $m \leq 26$
  - periodicity (typically) if $m \geq 27$

- "moderate" toll sequence: $c_n \sim n^\gamma L(n)$, $\frac{1}{2} < \gamma < 1$:
“small” toll sequence: $c_n = O(n^{1/2}L(n))$:
- asymptotic normality if $m \leq 26$
- periodicity (typically) if $m \geq 27$

“moderate” toll sequence: $c_n \sim n^{\gamma}L(n)$, $\frac{1}{2} < \gamma < 1$:
- convergence to non-normal distributions if $m \leq m_0$
  (where $m_0 \geq 26$)
“small” toll sequence: $c_n = O(n^{1/2}L(n))$:
- asymptotic normality if $m \leq 26$
- periodicity (typically) if $m \geq 27$

“moderate” toll sequence: $c_n \sim n^\gamma L(n), \quad \frac{1}{2} < \gamma < 1$:
- convergence to non-normal distributions if $m \leq m_0$
  (where $m_0 \geq 26$)
- periodicity (typically) if $m \geq m_0 + 1$
Distributional asymptotics & phase changes

- "small" toll sequence: \( c_n = O(n^{1/2}L(n)) \):
  - asymptotic normality if \( m \leq 26 \)
  - periodicity (typically) if \( m \geq 27 \)

- "moderate" toll sequence: \( c_n \sim n^\gamma L(n), \quad \frac{1}{2} < \gamma < 1 \):
  - convergence to non-normal distributions if \( m \leq m_0 \)
    (where \( m_0 \geq 26 \))
  - periodicity (typically) if \( m \geq m_0 + 1 \)

- "large" toll sequence: \( c_n \sim n^\gamma L(n) \) with \( \gamma \geq 1 \):
Distributional asymptotics & phase changes

- “small” toll sequence: \( c_n = O(n^{1/2}L(n)) \):
  - asymptotic normality if \( m \leq 26 \)
  - periodicity (typically) if \( m \geq 27 \)

- “moderate” toll sequence: \( c_n \sim n^\gamma L(n), \quad \frac{1}{2} < \gamma < 1 \):
  - convergence to non-normal distributions if \( m \leq m_0 \) (where \( m_0 \geq 26 \))
  - periodicity (typically) if \( m \geq m_0 + 1 \)

- “large” toll sequence: \( c_n \sim n^\gamma L(n) \) with \( \gamma \geq 1 \):
  - convergence to non-normal distributions for all values of \( m \) [\( \gamma = 1 \): Quicksort # of comparisons]
2. Moments: full asymptotic expansions

recall: moments \((a_n)\) of each specified order satisfy

\[
a_n = \frac{m}{m-1} \sum_j \binom{n-j-1}{m-2} a_j + b_n
\]

with \((b_n)\) defined in terms of moments of lower orders, and (for vanishing initial conditions)

\[
A(z) = B(z) + m! \sum_{j=1}^{m-1} \frac{(1-z)^{-\lambda_j}}{\psi'(\lambda_j)} \int_{\zeta=0}^{z} B(\zeta)(1-\zeta)^{\lambda_j-1} d\zeta
\]
Moment expansions

Remember

\[ A(z) = B(z) + m! \sum_{j=1}^{m-1} \frac{(1 - z)^{-\lambda_j}}{\psi'(\lambda_j)} \int_{\zeta=0}^{z} B(\zeta)(1 - \zeta)^{\lambda_j - 1} d\zeta \]
Moment expansions

- Remember

\[ A(z) = B(z) + m! \sum_{j=1}^{m-1} \frac{(1 - z)^{-\lambda_j}}{\psi'(\lambda_j)} \int_{\zeta=0}^{\zeta} B(\zeta)(1 - \zeta)^{\lambda_j-1} d\zeta \]

- Full asymptotic expansion for g.f. of toll sequence \((c_n)\)
  transfers to one for g.f. of sequence of means
Moment expansions

6 remember

\[ A(z) = B(z) + m! \sum_{j=1}^{m-1} \frac{(1 - z)^{-\lambda_j}}{\psi'(\lambda_j)} \int_{\zeta=0}^{z} B(\zeta)(1 - \zeta)^{\lambda_j-1} d\zeta \]

6 full asymptotic expansion for g.f. of toll sequence \((c_n)\) transfers to one for g.f. of sequence of means

6 inductively, get full asymptotic expansion for g.f. of sequence of moments of each order
Moment expansions

- Remember

\[ A(z) = B(z) + m! \sum_{j=1}^{m-1} \frac{(1 - z)^{-\lambda_j}}{\psi'(-\lambda_j)} \int_{\zeta=0}^{z} B(\zeta)(1 - \zeta)^{\lambda_j - 1} d\zeta \]

- Full asymptotic expansion for g.f. of toll sequence \((c_n)\) transfers to one for g.f. of sequence of means.

- Inductively, get full asymptotic expansion for g.f. of sequence of moments of each order.

- Use singularity analysis [Flajolet & Odlyzko, 1990] to get full asymptotic expansions for moments.
Prototype: the shape functional

\[ c_n := \ln \left( \binom{n}{m-1} \right), \quad n \geq m - 1 \]
Prototype: the shape functional

\[ c_n := \ln \left( \binom{n}{m-1} \right), \quad n \geq m - 1 \]

vanishing initial conditions
Prototype: the shape functional

$\ln \binom{n}{m-1}, n \geq m - 1$

vanishing initial conditions

using Tauberian methods, with $C_1 := K_1/(H_m - 1)$,
Prototype: the shape functional

\[ c_n := \ln \left( \binom{n}{m-1} \right), \quad n \geq m - 1 \]

vanishing initial conditions

using Tauberian methods, with \( C_1 := K_1/(H_m - 1) \),

first moments: \( \mu_n \sim C_1 n \)
Prototype: the shape functional

\[ c_n := \ln \left( \frac{n}{m-1} \right), \ n \geq m - 1 \]

vanishing initial conditions

using Tauberian methods, with \( C_1 := \frac{K_1}{(H_m - 1)} \),

- first moments: \( \mu_n \sim C_1 n \)
- second moments: \( \nu_n \sim C_1^2 n^2 \)
Prototype: the shape functional

\[ c_n := \ln \binom{n}{m-1}, \, n \geq m - 1 \]

\[ \text{vanishing initial conditions} \]

\[ \text{using Tauberian methods, with } C_1 := K_1/(H_m - 1), \]

\[ \begin{align*}
\Delta \text{ first moments: } & \mu_n \sim C_1 n \\
\Delta \text{ second moments: } & \nu_n \sim C_1^2 n^2 \\
\Delta \text{ can’t go beyond lead order, because “side conditions” can’t be established} & 
\end{align*} \]
Prototype: the shape functional

\[ c_n := \ln \left( \frac{n}{m-1} \right), \quad n \geq m - 1 \]

vanishing initial conditions

using Tauberian methods, with \( C_1 := \frac{K_1}{(H_m - 1)} \),

- first moments: \( \mu_n \sim C_1 n \)
- second moments: \( \nu_n \sim C_1^2 n^2 \)

- can’t go beyond lead order, because “side conditions” can’t be established
- in fact, can’t get lead order for variance
alternative approach: singularity analysis [Flajolet & Odlyzko, 1990]
alternative approach: singularity analysis [Flajolet & Odlyzko, 1990]

aids in determination of asymptotic order of growth of (Taylor) coefficients of functions analytic at the origin

\[ f(z) = O(g(z)) \implies f_n = O(g_n) \]
\[ f(z) = o(g(z)) \implies f_n = o(g_n) \]
\[ f(z) \sim g(z) \implies f_n \sim g_n. \]
alternative approach: **singularity analysis** [Flajolet & Odlyzko, 1990]

aids in determination of asymptotic order of growth of (Taylor) coefficients of functions analytic at the origin

\[ f(z) = O(g(z)) \Rightarrow f_n = O(g_n) \]

\[ f(z) = o(g(z)) \Rightarrow f_n = o(g_n) \]

\[ f(z) \sim g(z) \Rightarrow f_n \sim g_n. \]

based on Cauchy integral formula in “indented crown”, judicious choices of contours
alternative approach: **singularity analysis** [Flajolet & Odlyzko, 1990]

aids in determination of asymptotic order of growth of (Taylor) coefficients of functions analytic at the origin

\[
f(z) = O(g(z)) \implies f_n = O(g_n) \\
f(z) = o(g(z)) \implies f_n = o(g_n) \\
f(z) \sim g(z) \implies f_n \sim g_n.
\]

based on Cauchy integral formula in “indented crown”, judicious choices of contours

no side conditions on sequence required
Flajolet and Odlyzko [1990] give sufficient conditions for validity of these implications when $g(z)$ is of the form

$$(1 - z)^\beta (\log (1 - z))^\gamma (\log \log (1 - z))^\delta$$
Flajolet and Odlyzko [1990] give sufficient conditions for validity of these implications when \( g(z) \) is of the form
\[
(1 - z)^\beta (\log (1 - z))^{\gamma} (\log \log (1 - z))^{\delta}
\]

Flajolet [1998] showed that singularity analysis is applicable to the \textit{generalized polylogarithm}

\[
\text{Li}_{\alpha, r}(z) := \sum_{n=1}^{\infty} \frac{(\ln n)^r z^n}{n^{\alpha}}
\]
Recall

\[ A(z) = B(z) + m! \sum_{j=1}^{m-1} \frac{(1 - z)^{-\lambda_j}}{\psi'(\lambda_j)} \int_{\zeta=0}^{1} B(\zeta)(1 - \zeta)^{\lambda_j - 1} d\zeta \]

where \( B(z) \) is the g.f. of a sequence that depends on the toll sequence. We need
Recall

\[ A(z) = B(z) + m! \sum_{j=1}^{m-1} \frac{(1 - z)^{-\lambda_j}}{\psi'(\lambda_j)} \int_{\zeta=0}^{z} B(\zeta) (1 - \zeta)^{\lambda_j-1} d\zeta \]

where \( B(z) \) is the g.f. of a sequence that depends on the toll sequence. We need

- (1) closure under indefinite integration
Recall

\[ A(z) = B(z) + m! \sum_{j=1}^{m-1} \frac{(1-z)^{-\lambda_j}}{\psi'(\lambda_j)} \int_{\zeta=0}^{z} B(\zeta)(1-\zeta)^{\lambda_j-1} d\zeta \]

where \( B(z) \) is the g.f. of a sequence that depends on the toll sequence. We need

- (1) closure under indefinite integration
- (2) closure under differentiation (used, e.g., for closure under Hadamard product)
Recall

\[ A(z) = B(z) + m! \sum_{j=1}^{m-1} \frac{(1 - z)^{-\lambda_j}}{\psi'(\lambda_j)} \int_{\zeta=0}^{z} B(\zeta)(1 - \zeta)^{\lambda_j-1} d\zeta \]

where \( B(z) \) is the g.f. of a sequence that depends on the toll sequence. We need

- (1) closure under indefinite integration
- (2) closure under differentiation (used, e.g., for closure under Hadamard product)
- for example, \( f(z) = O((1 - z)^{\beta}) \) in indented crown implies \( f'(z) = O((1 - z)^{\beta-1}) \) in indented crown
(3) closure under Hadamard product:

\[(f \odot g)(z) := \sum_{n=0}^{\infty} f_n g_n z^n\]
(3) closure under Hadamard product:

\[(f \odot g)(z) := \sum_{n=0}^{\infty} f_n g_n z^n\]

these three closure results extend SA toolbox for asymptotic expansions
using the integral representation of the generating function of the mean and singularity analysis we can show (for $m$ even)

$$G_1(z) \sim \sum_{j=1}^{m-1} \tilde{A}_j (1 - z)^{-\lambda_j} + \tilde{B} - (1 - z)^{-1} \log (1 - z)^{-1}$$

$$+ L_{-1}(1 - z)^{-1} + \sum_{r=0}^{m-2} L_r (1 - z)^r$$

$$+ \sum_{r \geq m-1} [K_r (1 - z)^r \log (1 - z)^{-1} + L_r (1 - z)^r]$$
transfer back to get (for $m$ even)

$$
\mu_n \sim \sum_{j=1}^{m-1} \tilde{A}_j \frac{\langle \lambda_j \rangle_n}{n!} - H_n + L_{-1}
$$

$$
+ \sum_{r \geq m-1} K_r \Delta^r \left( \frac{1}{n} \right)
$$

$$
= (1 + o(1))C_1 n
$$
Application: shape functional (2)

transfer back to get (for $m$ even)

$$
\mu_n \sim \sum_{j=1}^{m-1} \tilde{A}_j \frac{\langle \lambda_j \rangle_n}{n!} - H_n + L_{-1} + \sum_{r \geq m-1} K_r \Delta^r \left( \left( \frac{1}{n} \right) \right)
$$

$$
= (1 + o(1))C_1 n
$$

similar result for $m$ odd except for an extra

$$
(1 - z)^m [\log (1 - z)^{-1}]^2
$$
term in the generating function
Application: second moments

Alternative form of solution to recurrence relation: for vanishing initial conditions (e.g.),

$$A(z) = \sum_{j=1}^{m-1} \frac{(1 - z)^{-\lambda_j}}{\psi'(\lambda_j)} \int_{\zeta=0}^{\zeta} B^{(m-1)}(\zeta)(1 - \zeta)^{\lambda_j + m - 2} d\zeta$$
Application: second moments

alternative form of solution to recurrence relation: for vanishing initial conditions (e.g.),

\[ A(z) = \sum_{j=1}^{m-1} \frac{(1 - z)^{-\lambda_j}}{\psi'(\lambda_j)} \int_{\zeta=0}^{z} B^{(m-1)}(\zeta)(1 - \zeta)^{\lambda_j + m - 2} d\zeta \]

when \( A(z) \) is the second-moment g.f. \( G_2(z) \),

\[ B^{(m-1)}(z) = (m - 1)m!(1 - z)^{-(m-2)} G_1^2(z) + D^{(m-1)}(z), \]

where

\[ d_n := c_n(2\mu_n - c_n), \quad n \geq m - 1 \]