

(C) The Kolmogorov 4/5 - law

We have focussed so far on the absolute structure functions, that were used to obtain bounds on the energy flux Π_ℓ . However, there are other types of structure functions of interest, some of them more directly related to energy flux and which, in fact, provide alternate definitions of it. Rather than define “large-scale energy” by

$$\bar{e}_\ell(\mathbf{x}, t) = \frac{1}{2} |\bar{\mathbf{v}}_\ell(\mathbf{x}, t)|^2$$

one can instead make an alternate definition by filtering just one factor

$$\begin{aligned} e_\ell(\mathbf{x}, t) &= \frac{1}{2} \mathbf{v}(\mathbf{x}, t) \cdot \bar{\mathbf{v}}_\ell(\mathbf{x}, t) \\ &= \int d^d \mathbf{r} G_\ell(\mathbf{r}) \frac{1}{2} \mathbf{v}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x} + \mathbf{r}, t) \end{aligned}$$

where

$$e_{\mathbf{r}}(\mathbf{x}, t) = \frac{1}{2} \mathbf{v}(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x} + \mathbf{r}, t)$$

is the so-called point-split kinetic energy density. The quantities $e_\ell(\mathbf{x}, t)$ and $e_{\mathbf{r}}(\mathbf{x}, t)$ are well-defined whenever \mathbf{v} has finite mean energy:

$$\frac{1}{T} \int_0^T dt \int_V d^d \mathbf{x} \frac{1}{2} |\mathbf{v}(\mathbf{x}, t)|^2 < +\infty.$$

We now derive a balance equation for the point-split kinetic energy density of a Navier-Stokes solution, as follows

$$\begin{aligned} \partial_t \left(\frac{1}{2} \mathbf{v} \cdot \mathbf{v}' \right) &+ \nabla \cdot \left[\left(\frac{1}{2} \mathbf{v} \cdot \mathbf{v}' \right) \mathbf{v} + \frac{1}{2} (p \mathbf{v}' + p' \mathbf{v}) + \frac{1}{4} |\mathbf{v}'|^2 \delta \mathbf{v} - \nu \nabla \left(\frac{1}{2} \mathbf{v} \cdot \mathbf{v}' \right) \right] \\ &= \frac{1}{4} \nabla_{\mathbf{r}} \cdot [\delta \mathbf{v} |\delta \mathbf{v}|^2] - \nu \nabla \mathbf{v} : \nabla \mathbf{v}' + \frac{1}{2} (\mathbf{f} \cdot \mathbf{v}' + \mathbf{f}' \cdot \mathbf{v}) \end{aligned}$$

with the notations

$$\begin{aligned} \mathbf{v} &= \mathbf{v}(\mathbf{x}, t), \quad p = p(\mathbf{x}, t) \\ \mathbf{v}' &= \mathbf{v}(\mathbf{x} + \mathbf{r}, t), \quad p' = p(\mathbf{x} + \mathbf{r}, t) \\ \delta \mathbf{v} &= \mathbf{v}' - \mathbf{v} = \mathbf{v}(\mathbf{x} + \mathbf{r}, t) - \mathbf{v}(\mathbf{x}, t) \end{aligned}$$

Remarks:

#1. This identity was first derived by L. Onsager in the 1940's in a smoothed and space-integrated form. The result was communicated to C. C. Lin in a letter in 1945, but never formally published by Onsager. The space-local form was derived by J. Duchan & R. Robert, Nonlinearity **13** 249-255(2000) in a smoothed version, discussed a bit later.

#2. The relation is analogous to the energy balance equation that we derived in the filtering approach:

$$\partial_t(\frac{1}{2}|\bar{\mathbf{v}}_\ell|^2) + \nabla \cdot \left[\frac{1}{2}|\bar{\mathbf{v}}_\ell|^2 \bar{\mathbf{v}}_\ell + \bar{p}_\ell \bar{\mathbf{v}}_\ell - \boldsymbol{\tau}_\ell \cdot \bar{\mathbf{v}}_\ell - \nu \nabla (\frac{1}{2}|\bar{\mathbf{v}}_\ell|^2) \right] = \nabla \bar{\mathbf{v}}_\ell : \boldsymbol{\tau}_\ell - \nu |\nabla \bar{\mathbf{v}}_\ell|^2 + \mathbf{f}_\ell \cdot \bar{\mathbf{v}}_\ell$$

Proof of the identity: Take

$$\begin{aligned} \partial_t \mathbf{v} + \nabla \cdot (\mathbf{v}\mathbf{v}) - \nu \Delta \mathbf{v} + \nabla p &= \mathbf{f}, \quad \nabla \cdot \mathbf{v} = 0 \\ \partial_t \mathbf{v}' + \nabla \cdot (\mathbf{v}'\mathbf{v}') - \nu \Delta \mathbf{v}' + \nabla p' &= \mathbf{f}', \quad \nabla \cdot \mathbf{v}' = 0 \end{aligned}$$

Dot the first by \mathbf{v}' and the second by \mathbf{v} and add together, to obtain

$$\begin{aligned} \partial_t(\mathbf{v} \cdot \mathbf{v}') + \mathbf{v} \cdot [\nabla \cdot (\mathbf{v}'\mathbf{v}') - \nu \Delta \mathbf{v}'] + \mathbf{v}' \cdot [\nabla \cdot (\mathbf{v}\mathbf{v}) - \nu \Delta \mathbf{v}] \\ + \nabla \cdot (p'\mathbf{v} + p\mathbf{v}') = \mathbf{f}' \cdot \mathbf{v} + \mathbf{f} \cdot \mathbf{v}' \end{aligned}$$

Now the viscous term is reorganized as

$$\begin{aligned} \mathbf{v} \cdot \Delta \mathbf{v}' + \mathbf{v}' \cdot \Delta \mathbf{v} &= \nabla \cdot [v_i \nabla v'_i + v'_i \nabla v_i] - 2 \nabla v_i \cdot \nabla v'_i \\ &= \nabla \cdot [\nabla(\mathbf{v} \cdot \mathbf{v}')] - 2 \nabla \mathbf{v} : \nabla \mathbf{v}' \end{aligned}$$

Lastly we discuss the crucial nonlinear term. Note first that

$$\begin{aligned} \mathbf{v} \cdot [\nabla \cdot (\mathbf{v}'\mathbf{v}')] + \mathbf{v}' \cdot [\nabla \cdot (\mathbf{v}\mathbf{v})] &= v_i \partial_j (v'_i v'_j) + v'_i \partial_j (v_i v_j) \\ &= v_i \partial_j (v'_i v'_j) + \partial_j (v'_i v_i v_j) - v_i v_j (\partial_j v'_i) \\ &= \Delta + \nabla \cdot [(\mathbf{v} \cdot \mathbf{v}')\mathbf{v}] \end{aligned}$$

with

$$\Delta \equiv v_i \partial_j (v'_i v'_j) - v_i v_j (\partial_j v'_i)$$

By incompressibility,

$$\begin{aligned}
\Delta &= v_i v'_j (\partial_j v'_i) - v_i v_j (\partial_j v'_i) \\
&= v_i (v'_j - v_j) (\partial_j v'_i) \\
&= v_i \delta v_j (\partial_j v'_i)
\end{aligned}$$

Also by incompressibility $\nabla_{\mathbf{r}} \cdot (\delta \mathbf{v}) = \nabla_{\mathbf{r}} \cdot \mathbf{v}' = 0$, so that

$$\begin{aligned}
\nabla_{\mathbf{r}} \cdot [\delta \mathbf{v} |\delta \mathbf{v}|^2] &= (\delta \mathbf{v} \cdot \nabla') |\delta \mathbf{v}|^2 \\
&= 2(\delta v_j \partial_j) v'_i \cdot (v'_i - v_i) \\
&= 2\delta v_j \cdot v'_i \partial_j v'_i - 2v_i \delta v_j (\partial_j v'_i) \\
&= \delta \mathbf{v} \cdot \nabla (|\mathbf{v}'|^2) - 2\Delta \\
&= \nabla \cdot [|\mathbf{v}'|^2 \delta \mathbf{v}] - 2\Delta
\end{aligned}$$

Thus,

$$\begin{aligned}
\partial_t (\mathbf{v} \cdot \mathbf{v}') &+ \nabla \cdot \left[(\mathbf{v} \cdot \mathbf{v}') \mathbf{v} + (p \mathbf{v}' + p' \mathbf{v}) + \frac{1}{2} |\mathbf{v}'|^2 \delta \mathbf{v} - \nu \nabla (\mathbf{v} \cdot \mathbf{v}') \right] \\
&- \frac{1}{2} \nabla_{\mathbf{r}} \cdot [\delta \mathbf{v} |\delta \mathbf{v}|^2] + 2\nu \nabla \mathbf{v} : \nabla \mathbf{v}' = (\mathbf{f} \cdot \mathbf{v}' + \mathbf{f}' \cdot \mathbf{v})
\end{aligned}$$

QED!

Multiplying the point-split identity through by $G_\ell(\mathbf{r})$ and integrating over \mathbf{r} gives a corresponding balance equation for the regularized energy density $\frac{1}{2} \mathbf{v} \cdot \bar{\mathbf{v}}_\ell$:

$$\begin{aligned}
\partial_t \left(\frac{1}{2} \mathbf{v} \cdot \bar{\mathbf{v}}_\ell \right) &+ \nabla \cdot \left[\left(\frac{1}{2} \mathbf{v} \cdot \bar{\mathbf{v}}_\ell \right) \mathbf{v} + \frac{1}{2} (p \bar{\mathbf{v}}_\ell + \bar{p}_\ell \mathbf{v}) + \frac{1}{4} \overline{(|\mathbf{v}|^2 \mathbf{v})}_\ell - \frac{1}{4} \overline{(|\mathbf{v}|^2)}_\ell \mathbf{v} - \nu \nabla \left(\frac{1}{2} \mathbf{v} \cdot \bar{\mathbf{v}}_\ell \right) \right] \\
&= -\frac{1}{4\ell} \int d^d \mathbf{r} (\nabla G)_\ell(\mathbf{r}) \cdot \delta \mathbf{v}(\mathbf{r}) |\delta \mathbf{v}(\mathbf{r})|^2 - \nu \nabla \mathbf{v} : \nabla \bar{\mathbf{v}}_\ell + \frac{1}{2} (\mathbf{f} \cdot \bar{\mathbf{v}}_\ell + \bar{\mathbf{f}}_\ell \cdot \mathbf{v})
\end{aligned}$$

The above equation can be shown to be valid even for singular Leray solutions of INS, if the space-time derivatives are interpreted in the sense of distributions. (See [Appendix](#).)

We now consider the limit of vanishing viscosity $\nu \rightarrow 0$. If the Navier-Stokes solution $\mathbf{v}^\nu \rightarrow \mathbf{v}$ as $\nu \rightarrow 0$ in the space-time L^2 -sense, i.e.

$$\int dt \int d^d \mathbf{x} |\mathbf{v}^\nu(\mathbf{x}, t) - \mathbf{v}(\mathbf{x}, t)|^2 \rightarrow 0, \text{ as } \nu \rightarrow 0$$

then it is not hard to show that the limiting velocity \mathbf{v} is a solution of the incompressible Euler equations

$$\partial_t \mathbf{v} + \nabla \cdot (\mathbf{v}\mathbf{v}) = -\nabla p + \mathbf{f}, \quad \nabla \cdot \mathbf{v} = 0$$

in the sense of space-time distributions. If it is furthermore true that $\int dt \int d^d \mathbf{x} |\mathbf{v}(\mathbf{x}, t)|^3 < +\infty$, then the point-split balance holds also for the Euler solution in the distribution sense:

$$\begin{aligned} \partial_t \left(\frac{1}{2} \mathbf{v} \cdot \bar{\mathbf{v}}_\ell \right) + \nabla \cdot \left[\left(\frac{1}{2} \mathbf{v} \cdot \bar{\mathbf{v}}_\ell \right) \mathbf{v} + \frac{1}{2} (p \bar{\mathbf{v}}_\ell + \bar{p}_\ell \mathbf{v}) + \frac{1}{4} \overline{(|\mathbf{v}|^2 \mathbf{v})}_\ell - \frac{1}{4} \overline{(|\mathbf{v}|^2)}_\ell \mathbf{v} \right] \\ = -\frac{1}{4\ell} \int d^d \mathbf{r} (\nabla G)_\ell(\mathbf{r}) \cdot \delta \mathbf{v}(\mathbf{r}) |\delta \mathbf{v}(\mathbf{r})|^2 \end{aligned} \quad (\star)$$

For simplicity, we consider the case with no external force, $\mathbf{f} = 0$. We now consider the limit $\ell \rightarrow 0$. Under the same basic assumption, that $\int dt \int d^d \mathbf{x} |\mathbf{v}(\mathbf{x}, t)|^3 < +\infty$, it is not hard to show that the LHS of equation (\star)

$$\begin{aligned} \partial_t \left(\frac{1}{2} \mathbf{v} \cdot \bar{\mathbf{v}}_\ell \right) + \nabla \cdot \left[\left(\frac{1}{2} \mathbf{v} \cdot \bar{\mathbf{v}}_\ell \right) \mathbf{v} + \frac{1}{2} (p \bar{\mathbf{v}}_\ell + \bar{p}_\ell \mathbf{v}) + \frac{1}{4} \overline{(|\mathbf{v}|^2 \mathbf{v})}_\ell - \frac{1}{4} \overline{(|\mathbf{v}|^2)}_\ell \mathbf{v} \right] \\ \rightarrow \partial_t \left(\frac{1}{2} |\mathbf{v}|^2 \right) + \nabla \cdot \left[\left(\frac{1}{2} |\mathbf{v}|^2 + p \right) \mathbf{v} \right] \equiv -D(\mathbf{v}), \quad \text{as } \ell \rightarrow 0 \end{aligned}$$

in the sense of distributions. Just to consider one typical term,

$$\begin{aligned} \left| \int dt \int d^d \mathbf{x} \nabla \varphi(\mathbf{x}, t) \cdot \left(\frac{1}{2} \mathbf{v}(\mathbf{x}, t) \cdot \bar{\mathbf{v}}_\ell(\mathbf{x}, t) \right) \mathbf{v}(\mathbf{x}, t) - \int dt \int d^d \mathbf{x} \nabla \varphi(\mathbf{x}, t) \cdot \mathbf{v}(\mathbf{x}, t) \frac{1}{2} |\mathbf{v}(\mathbf{x}, t)|^2 \right| \\ \leq \sup |\nabla \varphi| \cdot \left\| \left(\frac{1}{2} \mathbf{v} \cdot \bar{\mathbf{v}}_\ell \right) \mathbf{v} - \left(\frac{1}{2} |\mathbf{v}|^2 \right) \mathbf{v} \right\|_{L^1_{\text{spacetime}}} \end{aligned}$$

Using $\frac{1}{2} (\mathbf{v} \cdot \bar{\mathbf{v}}_\ell) \mathbf{v} - \frac{1}{2} |\mathbf{v}|^2 \mathbf{v} = \frac{1}{2} [\mathbf{v} \cdot (\bar{\mathbf{v}}_\ell - \mathbf{v})] \mathbf{v}$ and the Hölder inequality

$$\|[\mathbf{v} \cdot (\bar{\mathbf{v}}_\ell - \mathbf{v})] \mathbf{v}\|_{L^1} \leq \|\mathbf{v}\|_{L^3}^2 \|\bar{\mathbf{v}}_\ell - \mathbf{v}\|_{L^3}$$

One can then show that the upper bound $\rightarrow 0$ as $\ell \rightarrow 0$. This implies that

$$\nabla \cdot \left[\left(\frac{1}{2} \mathbf{v} \cdot \bar{\mathbf{v}}_\ell \right) \mathbf{v} \right] \rightarrow \nabla \cdot \left[\frac{1}{2} |\mathbf{v}|^2 \mathbf{v} \right]$$

in the sense of distributions. The other terms are treated in a similar fashion. But since the LHS of equation (\star) converges to $-D(\mathbf{v})$ in the sense of the distributions, so does the RHS!

That is,

$$D(\mathbf{v}) = \lim_{\ell \rightarrow 0} \frac{1}{4\ell} \int d^d \mathbf{r} (\nabla G)_\ell(\mathbf{r}) \cdot \delta \mathbf{v}(\mathbf{r}) |\delta \mathbf{v}(\mathbf{r})|^2 \quad (1)$$

in the sense of distributions. To summarize, we obtain the above formula for the anomalous dissipation $D(\mathbf{v})$ that appears in the energy balance relation

$$\partial_t \left(\frac{1}{2} |\mathbf{v}|^2 \right) + \nabla \cdot \left[\left(\frac{1}{2} |\mathbf{v}|^2 + p \right) \mathbf{v} \right] = -D(\mathbf{v}). \quad (2)$$

for the singular Euler solution $\mathbf{v}(\mathbf{x}, t)$. This result is quite interesting on its own rights and not just as a step in the proof of the Kolmogorov 4/5-law. It is a precise mathematical formulation of Onsager's idea that Euler solutions which arises in the zero-viscosity limit of turbulent flow may not conserve energy. We could derive the same balance equation (2) by starting from the balance equation for $\frac{1}{2} |\bar{\mathbf{v}}_\ell|^2$ and taking the limit $\ell \rightarrow 0$. This would give us another valid expression

$$D(\mathbf{v}) = \lim_{\ell \rightarrow 0} \Pi_\ell \quad (\text{in the distribution sense})$$

for the anomalous dissipation $D(\mathbf{v})$. In fact, the RHS of equation (\star)

$$D_\ell(\mathbf{v}) = \frac{1}{4\ell} \int d^d \mathbf{r} (\nabla G)_\ell(\mathbf{r}) \cdot \delta \mathbf{v}(\mathbf{r}) |\delta \mathbf{v}(\mathbf{r})|^2 \quad (3)$$

is another way of measuring energy flux to small scales, alternative to Π_ℓ . We can get from equation (1) for $D(\mathbf{v})$ Onsager's bound on energy flux to small scales. For example, if \mathbf{v} has Hölder exponent h , then it follows from equation (3) that

$$D_\ell(\mathbf{v}) = O(\ell^{3h-1})$$

the same bound derived earlier for Π_ℓ . These bounds imply the assertion of Onsager in his 1949 paper that Euler solutions must conserve energy if the velocity has Hölder exponent $h > 1/3$. Using L_p norms, one can easily show also that energy is conserved if $\sigma_p > \frac{1}{3}$ (equivalently, $\zeta_p > p/3$) for $p \geq 3$. Under any of these regularity assumptions, $D(\mathbf{v}) \equiv 0!$

Let us now return to our derivation of the 4/5-law, by obtaining a simplified expression for $D(\mathbf{v})$ for the case of a spherically symmetric filter kernel G that depends upon only the magnitude $r = |\mathbf{r}|$:

$$G(\mathbf{r}) = G(r)$$

so that

$$\nabla G(\mathbf{r}) = \hat{\mathbf{r}}G'(r).$$

In that case, one can go to spherical coordinates in d -dimensions

$$D_\ell(\mathbf{v}) = \frac{1}{4\ell} \int_0^\infty r^{d-1} dr \int_{S^{d-1}} d\omega(\hat{\mathbf{r}}) \hat{\mathbf{r}} \cdot \delta\mathbf{v}(\mathbf{r}) |\delta\mathbf{v}(\mathbf{r})|^2 (G')_\ell(r)$$

where S^{d-1} is the unit sphere in d -dimensions and $d\omega$ is the measure on solid angles. Now introduce

$$\delta v_L(\mathbf{r}) = \hat{\mathbf{r}} \cdot \delta\mathbf{v}(\mathbf{r}) = \underline{\text{longitudinal velocity increment}}$$

and

$$\begin{aligned} \langle \delta v_L(\mathbf{r}) |\delta\mathbf{v}(\mathbf{r})|^2 \rangle_{ang} &= \frac{1}{\Omega_{d-1}} \int_{S^{d-1}} d\omega(\hat{\mathbf{r}}) \delta v_L(\mathbf{r}) |\delta\mathbf{v}(\mathbf{r})|^2 \\ &= \text{angular average of } \delta v_L |\delta\mathbf{v}|^2 \end{aligned}$$

where Ω_{d-1} is the $(d-1)$ -dimension volume of S^{d-1} . We thus find that

$$\begin{aligned} D_\ell(\mathbf{v}) &= \frac{1}{4\ell} \Omega_{d-1} \int_0^\infty r^{d-1} dr (G')_\ell(r) \langle \delta v_L(\mathbf{r}) |\delta\mathbf{v}(\mathbf{r})|^2 \rangle_{ang} \\ &= \Omega_{d-1} \int_0^\infty \rho^d d\rho G'(\rho) \left. \frac{\langle \delta v_L(\mathbf{r}) |\delta\mathbf{v}(\mathbf{r})|^2 \rangle_{ang}}{4r} \right|_{\mathbf{r}=\ell\boldsymbol{\rho}} \end{aligned}$$

where $\boldsymbol{\rho} = \mathbf{r}/\ell$. We know that the limit of the LHS exists as $\ell \rightarrow 0$ in the sense of distributions and gives $D(\mathbf{v})$. Taking the limit on the RHS, we see that

$$\frac{\langle \delta v_L(\mathbf{r}) |\delta\mathbf{v}(\mathbf{r})|^2 \rangle_{ang}}{4r} \longrightarrow D^*(\mathbf{v}) \text{ , as } r \rightarrow 0$$

with

$$\begin{aligned} D(\mathbf{v}) &= D^*(\mathbf{v}) \cdot \Omega_{d-1} \int_0^\infty \rho^d d\rho G'(\rho) \\ &= D^*(\mathbf{v}) \cdot (-d \cdot \Omega_{d-1} \int_0^\infty \rho^{d-1} d\rho G(\rho)) \text{ by integration by parts} \\ &= -d \cdot D^*(\mathbf{v}) \text{ since } \Omega_{d-1} \int_0^\infty \rho^{d-1} d\rho G(\rho) = 1 \end{aligned}$$

We conclude finally that

$$\lim_{r \rightarrow 0} \frac{\langle \delta v_L(\mathbf{r}) |\delta \mathbf{v}(\mathbf{r})|^2 \rangle_{ang}}{r} = -\frac{4}{d} D(\mathbf{v})$$

The results in the above form were given in the paper

J. Duchon & R. Robert, “Inertial energy dissipation for weak solution of incompressible Euler and Navier-Stokes equations,” *Nonlinearity*, **13** 249-255(2000).

It is possible, by an elaboration of these arguments, to derive expressions for $D(\mathbf{v})$ that involve only $\delta v_L(\mathbf{r})$, or mixed expressions that involve $\delta v_L(\mathbf{r})$ and the transverse velocity increment

$$\delta \mathbf{v}_T(\mathbf{r}) = \delta \mathbf{v}(\mathbf{r}) - \delta v_L(\mathbf{r}) \hat{\mathbf{r}}$$

which satisfies $\hat{\mathbf{r}} \cdot \delta \mathbf{v}_T(\mathbf{r}) = 0$, or

$$\delta v_T^2(\mathbf{r}) = |\delta \mathbf{v}_T(\mathbf{r})|^2 / (d - 1),$$

the magnitude of the transverse velocity increment per component. These are, in d -dimensions,

$$\begin{aligned} \lim_{r \rightarrow 0} \frac{\langle \delta u_L^3(\mathbf{r}) \rangle_{ang}}{r} &= -\frac{12}{d(d+2)} D(\mathbf{v}) \\ \lim_{r \rightarrow 0} \frac{\langle \delta v_L(\mathbf{r}) \delta v_T^2(\mathbf{r}) \rangle_{ang}}{r} &= -\frac{4}{d(d+2)} D(\mathbf{v}) \end{aligned}$$

For the derivation, see G. L. Eyink, *Nonlinearity* **16**, 137-145(2003). The idea is to look at separate equations for point-split energy densities $\mathbf{v}(\mathbf{x}, t) \cdot \mathbf{v}_L(\mathbf{x} + \mathbf{r}, t)$, $\mathbf{v}(\mathbf{x}, t) \cdot \mathbf{v}_T(\mathbf{x} + \mathbf{r}, t)$.

Example: Burgers Equation

The above discussion has been a bit abstract, so that it is useful to consider a concrete example. All the previous results have exact analogies for singular/distributional solutions of the inviscid Burgers equation, which can be shown to satisfy the energy balance equation

$$\partial_t(\frac{1}{2}u^2) + \partial_x(\frac{1}{3}u^3) = -D(u)$$

with $D(u) = \lim_{\ell \rightarrow 0} \frac{1}{12\ell} \int_{-\infty}^{+\infty} dr (G')_\ell(r) \delta u^3(r)$

in the sense of distributions. Alternately,

$$\lim_{r \rightarrow 0} \frac{\langle \delta u^3(r) \rangle_{ang}}{|r|} = -12D(u)$$

where $\langle \delta u^3(r) \rangle_{ang} = \frac{1}{2}[\delta u^3(+|r|) + \delta u^3(-|r|)]$. For the Khokhlov sawtooth solution in the limit $\nu \rightarrow 0$ it is straightforward to calculate explicitly that, with $r > 0$

$$\begin{aligned} \langle \delta u^3(r) \rangle_{ang} &= \frac{1}{2}[(r/t + \Delta u)^3 - (r/t)^3] \chi_{[-r,0]}(x) \\ &\quad + \frac{1}{2}[(r/t)^3 - (r/t - \Delta u)^3] \chi_{[0,r]}(x) \end{aligned}$$

so that

$$\frac{\langle \delta u^3(r) \rangle_{ang}}{r} \longrightarrow [\frac{1}{2}(\Delta u)^3 + \frac{1}{2}(\Delta u)^3] \delta(x) = (\Delta u)^3 \delta(x) \text{ , as } r \rightarrow 0.$$

Notice that this is equal to $-12\varepsilon(x)$, where $\varepsilon(x) = \lim_{r \rightarrow 0} \nu(\partial_x u^\nu)^2$ is the distributional limit of the viscous dissipation in $u^\nu(x, t)$ as $\nu \rightarrow 0$.

A similar result can be obtained for the $\nu \rightarrow 0$ limit of Leray solutions of the Navier-Stokes equation. These satisfy a local energy balance of the form

$$\partial_t(\frac{1}{2}|\mathbf{v}^\nu|^2) + \nabla \cdot \left[(\frac{1}{2}|\mathbf{v}^\nu|^2 + p^\nu)\mathbf{v}^\nu - \nu \nabla(\frac{1}{2}|\mathbf{v}^\nu|^2) \right] = -\nu|\nabla \mathbf{v}^\nu|^2 \quad (\text{or, } \leq -\nu|\nabla \mathbf{v}^\nu|^2), \quad (\star)$$

For simplicity, we shall only consider the case where “=” holds above rather than “ \leq ”. (For the general case, see [Appendix](#).) Let us assume that $\mathbf{v}^\nu \rightarrow \mathbf{v}$ as $\nu \rightarrow 0$ in the L^3 -sense in spacetime, i.e.

$$\int dt \int d^d x |\mathbf{v}^\nu(\mathbf{x}, t) - \mathbf{v}(\mathbf{x}, t)|^3 \longrightarrow 0.$$

This is stronger than the L^2 -convergence assumed earlier, so that, again, the limiting velocity \mathbf{v} is an Euler solution in distribution sense. Furthermore, it is now possible to check that the LHS of equation (\star) above has the limit

$$\lim_{\nu \rightarrow 0} \partial_t(\frac{1}{2}|\mathbf{v}^\nu|^2) + \nabla \cdot \left[(\frac{1}{2}|\mathbf{v}^\nu|^2 + p^\nu)\mathbf{v}^\nu - \nu \nabla(\frac{1}{2}|\mathbf{v}^\nu|^2) \right] = \partial_t(\frac{1}{2}|\mathbf{v}|^2) + \nabla \cdot \left[(\frac{1}{2}|\mathbf{v}|^2 + p)\mathbf{v} \right]$$

in distribution sense. The argument is very similar to that which we gave earlier for the limit $\ell \rightarrow 0$. Furthermore, the limit is exactly the same, i.e. $-D(\mathbf{v})!$ Since the limits of the LHS and the RHS of equation (\star) must be the same, we obtain

$$D(\mathbf{v}) = \lim_{\nu \rightarrow 0} \nu |\nabla \mathbf{v}^\nu|^2 = \lim_{\nu \rightarrow 0} \varepsilon^\nu \quad (\text{Duchon \& Robert, 2000})$$

in the sense of distributions. Notice the RHS of the above expression is non-negative, so that its limit also must be:

$$D(\mathbf{v}) \geq 0$$

More precisely, $D(\mathbf{v})$ is a nonnegative distribution, which satisfies $\int d^d \mathbf{x} dt \varphi(\mathbf{x}, t) D(\mathbf{v})(\mathbf{x}, t) \geq 0$ for every nonnegative test function φ (C^∞ with compact support). It is known that every nonnegative distribution is given by a measure, i.e.

$$\int d^d \mathbf{x} \int dt \varphi(\mathbf{x}, t) D(\mathbf{v})(\mathbf{x}, t) = \int \int \mu(d\mathbf{x}, dt) \varphi(\mathbf{x}, t)$$

This “dissipation measure” has been much studied experimentally and observed to have multi-fractal scaling properties, as we discuss a bit later! If φ is nonnegative and also normalized

$$\int dt \int d^d \mathbf{x} \varphi(\mathbf{x}, t) = 1,$$

then we can interpret

$$\int dt \int d^d \mathbf{x} \varphi(\mathbf{x}, t) \nu |\nabla \mathbf{v}^\nu(\mathbf{x}, t)|^2 \equiv \langle \nu |\nabla \mathbf{v}^\nu|^2 \rangle_\varphi$$

as an average in spacetime over the compact support of φ , weighted by φ . The above result then says that

$$\lim_{\nu \rightarrow 0} \langle \varepsilon^\nu \rangle_\varphi = \langle D(\mathbf{v}) \rangle_\varphi$$

Our earlier results can be stated in a similar fashion, e.g.

$$\lim_{r \rightarrow 0} \lim_{\nu \rightarrow 0} \frac{\langle [\delta v_L^\nu(\mathbf{r})]^3 \rangle_{\varphi, ang}}{r} = -\frac{12}{d(d+2)} \langle D(\mathbf{v}) \rangle_\varphi.$$

We thus see that, taking first $\nu \rightarrow 0$,

$$\langle \delta u_L^3(\mathbf{r}) \rangle_{\varphi, ang} \sim -\frac{12}{d(d+2)} \langle \varepsilon \rangle_\varphi r$$

This is the famous Kolmogorov 4/5-law (since the coefficient $\frac{12}{d(d+2)} = \frac{4}{5}$ for $d = 3$), derived by Kolmogorov in the third of his celebrated 1941 papers on turbulence. The related results

$$\begin{aligned} \langle \delta v_L(\mathbf{r}) \delta u_T^2(\mathbf{r}) \rangle_{\varphi, ang} &\sim -\frac{4}{d(d+2)} \langle \varepsilon \rangle_{\varphi} r \\ \langle \delta v_L(\mathbf{r}) |\delta \mathbf{v}(\mathbf{r})|^2 \rangle_{\varphi, ang} &\sim -\frac{4}{d} \langle \varepsilon \rangle_{\varphi} r \end{aligned}$$

are called the Kolmogorov 4/15- and 4/3-laws, respectively. These were derived by Kolmogorov in the statistical sense, averaging over an ensemble of solutions assuming statistical homogeneity and isotropy. He employed in his derivation an equation derived earlier for the 2-point velocity correlation $\langle v_i(\mathbf{x}, t) v_j(\mathbf{x} + \mathbf{r}, t) \rangle$ by van Kármán and Howarth (1938), so that this is sometimes called the Kolmogorov-Kármán-Howarth relation. The result presented here (following largely the derivation of Duchon & Robert, 2000) is much stronger, because there is no average over ensembles and no assumption of homogeneity and/or isotropy. It seems to have been Onsager in the 1940's who realized that such relations should hold for individual realizations, without averaging. He derived the formula

$$D_{\ell}(\mathbf{v}) = \frac{1}{4\ell} \int d^d r (\nabla G)_{\ell}(\mathbf{r}) \cdot \delta \mathbf{v}(\mathbf{r}) |\delta \mathbf{v}(\mathbf{r})|^2$$

and discussed its limit for $\ell \rightarrow 0$. In the statistical framework, the corresponding result

$$\nabla_{\mathbf{r}} \cdot \langle \delta \mathbf{v}(\mathbf{r}) |\delta \mathbf{v}(\mathbf{r})|^2 \rangle \sim -4 \langle \varepsilon \rangle, \text{ as } r \rightarrow 0$$

was derived by A. S. Monin (1959) and is sometimes called the Kolmogorov-Monin relation. It does not assume isotropy. There is another derivation of the 4/5-law by Nie & Tanveer (1999) without statistical averaging. It uses also spacetime averaging and angle-averaging. It is stronger than the result presented here in that it includes viscous corrections, but it is weaker than the presented local results, since it requires a global spacetime average.

Experiments and Simulations

★ K. Sreenivasan et al., PRL, vol.77, 1488-1491 (1996)

These authors present data for the 4/5-law obtained from experimental observations of the centerline of flow through a pipe at bulk Reynolds number $Re = 230,000$, and from a 512^3 DNS of homogenous forced turbulence at $Re_\lambda = 220 (\doteq \sqrt{Re})$. The data for velocity increments were not angle-averaged, so this is also a test of return to isotropy at small scales.

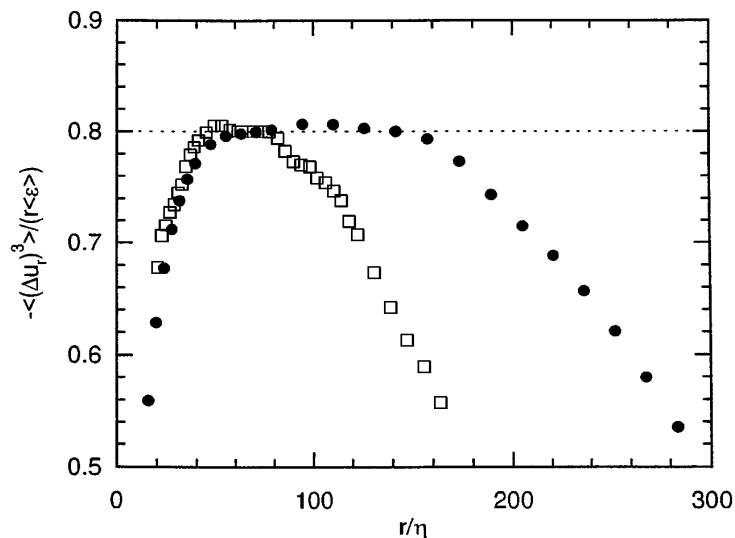


FIG. 2. The quantity $\langle \Delta u_r^3 \rangle / r$ as a function of r . Squares, experiment; circles, simulations; dots indicate Kolmogorov's $\frac{4}{5}$ th law. It is believed that the slight bump in the left part of the experimental data is the bottleneck effect [see G. Falkovich, Phys. Fluids **6**, 1411 (1994); D. Lohse and A. Mueller-Groeling, Phys. Rev. Lett. **74**, 1747 (1995)]. While the bottleneck effects discussed in these two papers refer especially to second-order structure functions (or to energy spectrum), a similar effect is likely to exist for the third-order as well. This is typical of most measurements [see, for example, Y. Gagne, Docteur ès-Sciences Physiques Thèse, Université de Grenoble, France (1987)].

★ T. Gotoh et al. Phys. Fluids **14**, 1065-1081 (2002)

This paper presents data from numerical simulations up to 1024^3 resolution of homogenous forced turbulence, and Re_λ in the range 38 – 460. Tests were made of both the 4/5-law and the 4/3-law, again without angle-averaging.

Velocity field statistics in homogeneous 1071

1072 Phys. Fluids, Vol. 14, No. 3, March 2002

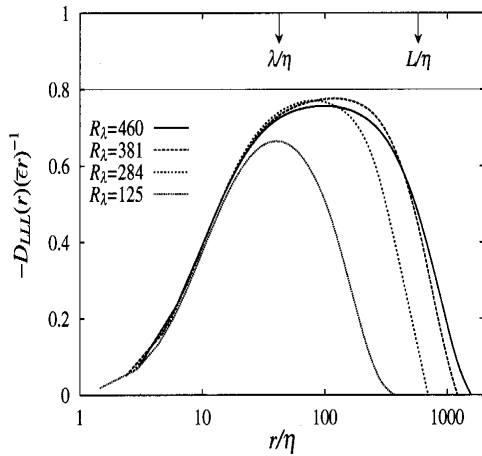


FIG. 12. Kolmogorov's 4/5 law. L/η and λ/η are shown for $R_\lambda=460$. The maximum values of the curves are 0.665, 0.771, 0.781, and 0.757 for $R_\lambda=125, 284, 381,$ and 460, respectively.

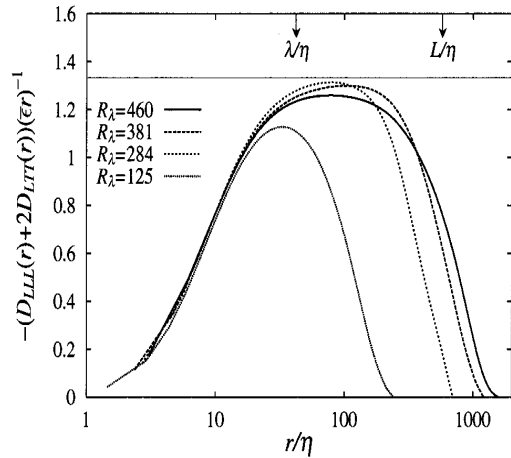


FIG. 14. Kolmogorov's 4/3 law. L/η and λ/η are shown for $R_\lambda=460$. The maximum values of the curves for the 4/3 law are 0.564, 1.313, 1.297, and 1.259 for $R_\lambda=125, 284, 381,$ and 460, respectively.

★ M. Taylor et al. Phys. Rev. E **68**, 026310 (2003)

Another numerical study in a 512^3 DNS of homogenous forced turbulence, with $Re_\lambda \cong 249 - 263$. This paper studies the effects of angle-averaging and obtains results with such averaging comparable to those of Gotoh et al.(2002) at nearly twice the Reynolds number.

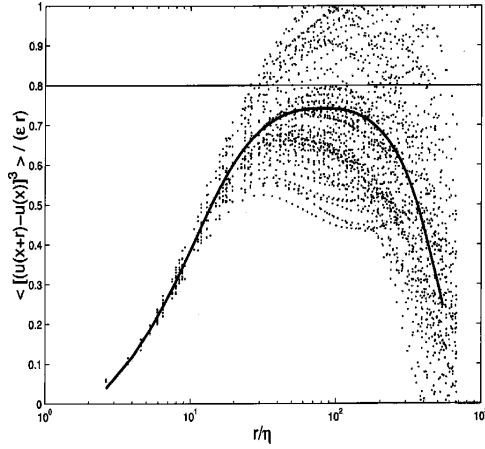


FIG. 6. The nondimensional third-order longitudinal structure function, computed from a single snapshot of the stochastic dataset, vs the nondimensional scale r/η . The dots indicate the values of the structure function computed at various ℓr_j . The thick curve is the angle average. The horizontal line indicates the $4/5$ mark.

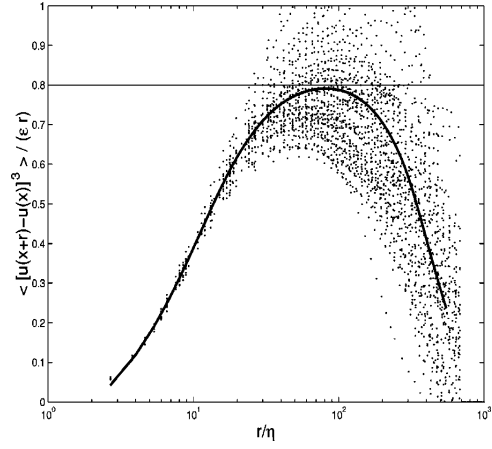


FIG. 7. The nondimensional third-order longitudinal structure function computed from a single snapshot of the deterministic dataset vs the nondimensional scale r/η . The various symbols and lines mean the same as in Fig. 6.

One last remark: The derivation that we have given applies even if $D(\mathbf{v}) \equiv 0$, i.e. vanishes everywhere. For example, this holds in a smooth solution of the Euler equations, for which $\langle \delta u_L^3(\mathbf{r}) \rangle_{ang} \sim \langle \delta v_L(\mathbf{r}) \delta \mathbf{v}_T^2(\mathbf{r}) \rangle \sim O(r^3)$, so that $D_\ell(\mathbf{v}) = O(\ell^2) \rightarrow 0$ as $\ell \rightarrow 0$. Another example is 2D Euler solutions where, under very general assumptions, $D(\mathbf{v}) \equiv 0$ and there is no energy cascade to small scales. E.g. see Proposition 6 in Duchon & Robert (2000). There is a nontrivial extension of the $\frac{4}{5}$ -law to 2D turbulence, but with $\langle \delta u_L^3(\mathbf{r}) \rangle_{ang}$ positive, corresponding to inverse energy cascade. E.g. see D. Bernard (1999).

Additional References:

T. von Kármán & L. Howarth, “On the statistical theory of isotropic turbulence,” Proc. Roy. Soc. Lond. A **164**, 192-215 (1938).

A. N. Kolmogorov, “Dissipation of energy in locally isotropic turbulence,” Dokl. Akad. Nauk. SSR **32**, 16-18 (1941).

A. S. Monin, "Theory of locally isotropic turbulence," Dokl. Akad. Nauk. SSSR **125** 515-518(1959); see also, A.S. Monin & A. M. Yaglom, Statistical Fluid Mechanics, vol.2 (MIT, 1975), p.403.

Papers on Deterministic Versions of the $\frac{4}{5}$ -law

Q. Nie & S. Tanveer, "A note on the third-order structure functions in turbulence," Proc. R. Soc. A **455**, 1615-1635(1999).

J. Duchon & Robert, "Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes," Nonlinearity, **13** 249-255(2000)

G. L. Eyink, "Local $\frac{4}{5}$ -law and energy dissipation anomaly in turbulence," Nonlinearity, **16** 137-145(2003).

G. L. Eyink, "Onsager and the theory of hydrodynamic turbulence," Rev. Mod. Phys. **78** 87-135(2006), Section IV, B.

2D Analogues of the $\frac{4}{5}$ -law

D. Bernard, "Three-point velocity correlation functions in two-dimensional forced turbulence," Phys. Rev. E **60** 6184-6187(1993).

A. M. Polyakov, "The theory of turbulence in two dimensions," Nucl. Phys. B **396**, 367-385(1993). This paper, in particular, discusses the analogy of the $\frac{4}{5}$ -law with conservation-law anomalies in quantum field theories.

APPENDIX I: LERAY SOLUTIONS OF INCOMPRESSIBLE NAVIER-STOKES

The purpose of this appendix is to discuss more carefully the issue of energy balance/energy dissipation for Leray Solutions of the incompressible Navier-Stokes (INS) equations, which are possibly singular. It was proved by

J. Leray, “Essai sur Le Mouvement d’un fluide visqueux emplissant l’espace,”

Acta. Math. **63** 193-248 (1934)

that solutions of the INS

$$\partial_t \mathbf{v}^\nu + \nabla \cdot (\mathbf{v}^\nu \mathbf{v}^\nu) = -\nabla p^\nu + \nu \Delta \mathbf{v}^\nu, \quad \nabla \cdot \mathbf{v}^\nu = 0$$

always exist in the sense of spacetime distributions, i.e.

$$\begin{aligned} & \int_{t_0}^T dt \int d^d \mathbf{x} [\partial_t \varphi(\mathbf{x}, t) \cdot \mathbf{v}^\nu(\mathbf{x}, t) + \nabla \varphi(\mathbf{x}, t) : \mathbf{v}^\nu(\mathbf{x}, t) \otimes \mathbf{v}^\nu(\mathbf{x}, t) \\ & + (\nabla \cdot \varphi(\mathbf{x}, t)) p^\nu(\mathbf{x}, t) - \nu \Delta \varphi(\mathbf{x}, t) \cdot \mathbf{v}^\nu(\mathbf{x}, t)] = \int d^d \mathbf{x} \varphi(\mathbf{x}, t_0) \cdot \mathbf{v}_0^\nu(\mathbf{x}, t), \\ & \int_{t_0}^T dt \int d^d \mathbf{x} \nabla \psi(\mathbf{x}, t) \cdot \mathbf{v}^\nu(\mathbf{x}, t) = 0 \end{aligned}$$

for any initial condition \mathbf{v}_0^ν with finite energy: $\|\mathbf{v}_0^\nu\|_{L^2(V)} < +\infty$. Here $\varphi = (\varphi_1, \dots, \varphi_d)$, ψ are C^∞ functions on spacetime with compact support. Note that the above rather abstract-looking formulation is actually very physical! It is equivalent to the following balance equation for the momentum in a bounded region $\Omega \subseteq V$ with smooth boundary $\partial\Omega$ and times $t > t' \geq t_0$

$$\begin{aligned} & \int_{\Omega} \mathbf{v}^\nu(\mathbf{x}, t) d^d \mathbf{x} - \int_{\Omega} \mathbf{v}^\nu(\mathbf{x}, t') d^d \mathbf{x} \\ & = - \int_{t'}^t d\tau \int_{\partial\Omega} \left[\mathbf{v}^\nu(\mathbf{x}, \tau) (\mathbf{v}^\nu(\mathbf{x}, \tau) \cdot \hat{\mathbf{n}}) + p^\nu(\mathbf{x}, \tau) \hat{\mathbf{n}} - \nu \frac{\partial \mathbf{v}^\nu}{\partial n}(\mathbf{x}, \tau) \right] dA, \\ & \int_{\partial\Omega} \mathbf{v}^\nu(\mathbf{x}, t) \cdot \hat{\mathbf{n}} dA = \int_{\partial\Omega} \mathbf{v}^\nu(\mathbf{x}, t') \cdot \hat{\mathbf{n}} dA = 0 \end{aligned}$$

for all possible choices of Ω, t, t' . These equations just state that the momentum change in a bounded region Ω comes from flux of momentum across the surface $\partial\Omega$ and that there is no net flux of mass across the surface.

In addition, Leray showed that his distributional solutions satisfy the following fundamental energy inequality for all $t \geq t_0$

$$\frac{1}{2} \|\mathbf{v}^\nu(t)\|_{L^2(V)}^2 + \nu \int_{t_0}^t d\tau \|\nabla \mathbf{v}^\nu(\tau)\|_{L^2(V)}^2 \leq \frac{1}{2} \|\mathbf{v}_0^\nu\|_{L^2(V)}^2$$

This inequality states that the total energy at time t plus the integrated energy dissipation up to time t must be less than or equal to the initial energy $\frac{1}{2} \|\mathbf{v}_0^\nu\|_{L^2(V)}^2$. If the solutions are everywhere smooth (“strong”), then the inequality “ \leq ” becomes equality “ $=$ ”. However, if the solutions are singular (“weak”), then there may be strict inequality, which corresponds to “extra dissipation” due to the singularities, in addition to the viscous dissipation. Note that it is a consequence of this fundamental energy inequality that

$$\begin{aligned} \sup_{t_0 \leq t \leq T} \|\mathbf{v}^\nu(t)\|_{L^2(V)} &\leq \|\mathbf{v}_0\|_{L^2(V)} < +\infty \\ \int_{t_0}^T dt \|\nabla \mathbf{v}^\nu(t)\|_{L^2(V)}^2 &\leq \frac{\|\mathbf{v}_0\|_{L^2(V)}^2}{2\nu} < +\infty \end{aligned}$$

for a fixed initial condition \mathbf{v}_0 with finite energy. From these estimates, some other basic bounds follow, such as

$$\int_{t_0}^T dt \|\mathbf{v}^\nu(t)\|_{L^3(V)}^3 \leq (\text{const.}) \frac{L^2}{\nu} \|\mathbf{v}_0\|_{L^2(V)}^3.$$

This inequality means that Leray solutions are sufficiently regular that one can consider a local energy balance in spacetime. To construct his solutions, Leray considered the limit as $\ell \rightarrow 0$ of the modified equation

$$\partial_t \mathbf{v} + (\bar{\mathbf{v}}_\ell \cdot \nabla) \mathbf{v} = -\nabla p + \nu \Delta \mathbf{v}$$

where only the velocity $\bar{\mathbf{v}}_\ell = G_\ell * \mathbf{v}$ appearing in the advection term has been smoothed. Leray showed that the above equation has regular solutions $(\mathbf{v}_\ell^*, p_\ell^*)$, which lie in a bounded (and thus weakly compact) subset of $L^2([0, T], H^1(V))$. Hence, weak limits $\mathbf{v}_\ell^* \rightarrow \mathbf{v}$ exist in this space, along subsequences of ℓ . Such \mathbf{v} can be seen to be distributional solutions of the incompressible Navier-Stokes equation

$$\partial_t \mathbf{v} + \nabla \cdot (\mathbf{v}\mathbf{v}) = -\nabla p + \nu \Delta \mathbf{v}$$

and, since

$$\lim_{\ell \rightarrow 0} \int d^d x \int dt |\nabla \mathbf{v}_\ell^*(\mathbf{x}, t)|^2 \psi(\mathbf{x}, t) \geq \int d^d x \int dt |\nabla \mathbf{v}(\mathbf{x}, t)|^2 \psi(\mathbf{x}, t)$$

for non-negative test functions ψ , also obeys the energy balance

$$\partial_t(\frac{1}{2}|\mathbf{v}|^2) + \nabla \cdot [(\frac{1}{2}|\mathbf{v}|^2 + p)\mathbf{v} - \nu \nabla(\frac{1}{2}|\mathbf{v}|^2)] = -D(\mathbf{v}) - \nu |\nabla \mathbf{v}|^2$$

with $D(\mathbf{v}) \geq 0$. By smoothing the solution \mathbf{v} one can write an energy balance also for $\bar{\mathbf{v}}_\ell$:

$$\begin{aligned} \partial_t(\frac{1}{2}\mathbf{v} \cdot \bar{\mathbf{v}}_\ell) &+ \nabla \cdot [(\frac{1}{2}\mathbf{v} \cdot \bar{\mathbf{v}}_\ell)\mathbf{v} + \frac{1}{2}(p\bar{\mathbf{v}}_\ell + \bar{p}_\ell\mathbf{v}) \\ &+ \frac{1}{4}(\overline{(|\mathbf{v}|^2\mathbf{v})}_\ell - \frac{1}{4}\overline{(|\mathbf{v}|^2)}\mathbf{v} - \nu \nabla \frac{1}{2}\mathbf{v} \cdot \bar{\mathbf{v}}_\ell)] \\ &= -\frac{1}{4\ell} \int d^d \mathbf{r} \nabla G_\ell(\mathbf{r}) \cdot \delta \mathbf{v}(\mathbf{r}) |\delta \mathbf{v}(\mathbf{r})|^2 - \nu \nabla \mathbf{v} : \nabla \bar{\mathbf{v}}_\ell \end{aligned}$$

Taking the limit $\ell \rightarrow 0$, we recover the previous energy balance with an explicit expression for $D(\mathbf{v})$:

$$D(\mathbf{v}) = \lim_{\ell \rightarrow 0} \frac{1}{4\ell} \int d^d \mathbf{r} (\nabla G)_\ell(\mathbf{r}) \cdot \delta \mathbf{v}(\mathbf{r}) |\delta \mathbf{v}(\mathbf{r})|^2$$

This formula makes it clear that $D(\mathbf{v})$ in the Navier-Stokes solution is connected with velocity increments. Note that, in general, in the presence of such singularities, the total Navier-Stokes dissipation is

$$D(\mathbf{v}^\nu) + \nu |\nabla \mathbf{v}^\nu|^2$$

(where we have now added the superscript ν to indicate the viscosity value) and in the $\nu \rightarrow 0$ limit it is this total dissipation which converges to the anomalous dissipation in the Euler solution

$$D(\mathbf{v}^\nu) + \nu |\nabla \mathbf{v}^\nu|^2 \rightarrow D(\mathbf{v}).$$

For more details, see Duchon & Robert (2000)

General References on Leray Solutions of INS

An English translation of Leray's paper of 1934 has been made by mathematician Bob Terrell:

<https://arxiv.org/abs/1604.02484>

A key paper on "partial regularity" of Leray solutions, improving the earlier work of V. Scheffer (1977) is

L. Caffarelli, R. Kohn & L. Nirenberg, "Partial regularity of suitable weak solutions of the Navier-Stokes equations," *Commun. Pure Appl. Math.* **35**, 771-831 (1982)

Many good textbooks presentation of the Leray theory are available. I'd recommend

R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis* (AMS, 2001)

and

R. Temam, *Navier-Stokes Equation & Nonlinear Functional Analysis* (SIAM, 1995)

especially for people interested in computational fluid mechanics. A good book for physicists is

G. Gallavotti, *Foundations of Fluid Dynamics*, Volume 172, Issue 5998 of *Theoretical and Mathematical Physics* (Springer Science & Business Media, 2002)

APPENDIX II: THE “ONLAGER CONJECTURE” AND h -PRINCIPLE

What is now called the “Onsager conjecture” goes back to the following remark that Onsager made at the very end of his 1949 paper on fluid turbulence:

“It is of some interest to note that in principle, turbulent dissipation as described could take place just as readily without the final assistance by viscosity. In the absence of viscosity, the standard proof of the conservation of energy does not apply, because the velocity field does not remain differentiable! In fact it is possible to show that the velocity field in such ‘ideal’ turbulence cannot obey any LIPSCHITZ condition of the form

$$(26) \quad |\vec{v}(\vec{r}' + \vec{r}') - \vec{v}(\vec{r}')| < (\text{const.})r^n$$

for any order n greater than $1/3$; otherwise the energy is conserved. Of course, under the circumstances, the ordinary formulation of the laws of motion in terms of differential equations becomes inadequate and must be replaced by a more general description; for example, the formulation (15) in terms of FOURIER series will do. The detailed conservation of energy (17) does not imply conservation of the total energy if the number of steps in the cascade is infinite, as expected and the double sum of $Q(\vec{k}, \vec{k}')$ converges only conditionally.”

First, Onsager claims here that “it is possible to show that” that energy is conserved by ideal (Euler) fluid equations if the Hölder exponent of the velocity is greater than $1/3$ and he gives a brief sketch of a proof using Fourier series. Second, Onsager remarks, after discussing the K41 theory in the preceding paragraphs, that “in principle, turbulent dissipation as described could take place just as readily without the final assistance by viscosity”, so that he clearly believed that Euler solutions with Hölder exponents $1/3$ (or smaller) could dissipate kinetic energy.

The first published proof of Onsager’s claim of conservation of kinetic energy for exponents $> 1/3$ was given in the paper:

G. L. Eyink, “Energy dissipation without viscosity in ideal hydrodynamics, I. Fourier analysis and local energy transfer,” *Physica D* **78** 222–240 (1994)

which made rigorous Onsager’s heuristic argument using Fourier series but which required a more restrictive notion than standard Hölder continuity. The paper

P. Constantin, W. E., and E.S. Titi, “Onsager’s conjecture on the energy conservation for solutions of Euler’s equation”, *Commun. Math. Phys.* **165** 207–209 (1994)

very concisely proved Onsager’s original statement on energy conservation for Euler solutions with velocities in Hölder spaces but also proved conservation for velocities only in Besov spaces. Their argument employed as a regularization a spatial mollification/low-pass filtering/coarse-graining operation, which is a very general and powerful technique that we have exploited throughout these lectures. Another important paper was

J. Duchon and R. Robert, “Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations,” *Nonlinearity* **13** 249–255 (2000)

which proved the Besov-space result using a point-splitting regularization that makes a connection with the Kolmogorov 4/5th law. Remarkably, Onsager had performed calculations identical to those of Duchon-Robert in his unpublished notes:

<https://ntnu.tind.io/record/121183>

see pp.14-18 in that folder. These results were never formally published but only communicated in a private letter to T. von Kármán and C. C. Lin in 1945, which is reproduced and discussed further here:

K. R. Sreenivasan, “Onsager and the theory of hydrodynamic turbulence,” *Rev. Mod. Phys.* **78** 87–135 (2006)

The second remark of Onsager, that energy dissipation should be possible for Euler solutions whose velocities have Hölder regularity $\leq 1/3$, was almost certainly not proved by him in any mathematical way but instead presumably suggested by the empirical evidence. The modern form of Onsager’s conjecture in terms of “weak solutions” was stated in Eyink (1994). That paper also constructed a simple example of a time-independent velocity field which showed that the paper’s own proof of conservation (and also that of Constantin-E-Titi) could not be extended to regularity exponents $\leq 1/3$. Eyink (1994) then noted that

“It must not, of course, be concluded that, simply because our argument fails when $h \leq 1/3$, that non-conservation is actually possible for $h \leq 1/3$. We emphasize that to demonstrate this it is necessary to construct an appropriate solution $\mathbf{v}(\cdot, \cdot)$ with $\mathbf{v}(\cdot, t) \in C^h$, $0 < h < 1/3$ for $t \in [0, T]$, for which the energy indeed decreases or increases in the interval.”

This issue then lay dormant until 13 years later when preprints appeared by Camillo De Lellis and László Székelyhidi Jr

C. De Lellis and L. Székelyhidi Jr, “The Euler equations as a differential inclusion,” arXiv:math/0702079 (2007) [published in *Annals of Mathematics*, **170** 1417–1436 (2009)]

C. De Lellis and L. Székelyhidi Jr, “On admissibility criteria for weak solutions of the Euler equations”, arXiv 0712.3288 (2007) [published in *Arch. Ration. Mech. Anal.* **195** 225–260 (2010)]

which initiated a long effort that culminated in full proofs in 2016-2017 that dissipative Euler solutions exist with any Hölder exponent $h < 1/3$:

P. Isett, “A proof of Onsager’s conjecture,” arXiv:1608.08301 (2016) [published in *Annals Math.* **188** 1–93 (2018)]

T. Buckmaster, C. De Lellis, L. Székelyhidi Jr. and V. Vicol, “Onsager’s conjecture for admissible weak solutions,” arXiv:1701.08678 (2017) [*Commun. Pure Appl. Math.* (in press, 2018)]

Remarkably, these developments are very closely connected with famous work of the mathematician John Nash on C^1 isometric embeddings:

J. Nash, “ C^1 isometric imbeddings”, *Ann. of Math.* **60** 383–396 (1954),

a connection which is very lucidly discussed in the following review papers:

C. De Lellis and L. Székelyhidi, Jr., “Continuous Dissipative Euler Flows and a Conjecture of Onsager,” in *Proceedings of the European Congress of Mathematics, Kraków, 2012*, edited by R. Latała, A. Ruciński, P. Strzelecki, J. Świątkowski, and D. Wrzosek (European Mathematical Society, Zurich, 2013), pp. 13–30.

http://cvgmt.sns.it/media/doc/paper/2187/de_lellis_proc_ECM_4.pdf

C. De Lellis and L. Székelyhidi, Jr., “High dimensionality and h-principle in PDE,” *Bull. Amer. Math. Soc.* **54** 247–282 (2017)

We give here a very succinct review, following the discussions in the previous references.

The paper of Nash (1954) addressed a classical problem of differential geometry, whether a smooth manifold M of dimension $n \geq 2$ with Riemannian metric g may be isometrically imbedded in m -dimensional Euclidean space \mathbb{R}^m , that is, whether a C^1 embedding map $\mathbf{u} : M \rightarrow \mathbb{R}^m$ exists so that the Riemannian metric induced by the embedding agrees with g , or

$$\partial_i \mathbf{u} \cdot \partial_j \mathbf{u} = g_{ij}. \quad (*)$$

To answer this question, Nash considered a more general problem of *short embeddings* which do not preserve lengths of curves on M but can only *decrease* lengths, so that

$$\partial_i \bar{\mathbf{u}} \cdot \partial_j \bar{\mathbf{u}} \leq g_{ij}$$

in the matrix sense. The startling result obtained by Nash, with a minor improvement due to

N. H. Kuiper, On C^1 -isometric imbeddings. I, II, Nederl. Akad. Wetensch. Proc. Ser. A. 58 = Indag. Math. 17 (1955), 545–556, 683–689.

is the following:

Nash-Kuiper Theorem: Let (M, g) be a smooth closed n -dimensional Riemannian manifold, and let $\bar{\mathbf{u}} : M \rightarrow \mathbb{R}^m$ be a C^∞ short embedding with $m \geq n + 1$. For any $\epsilon > 0$ there exists a C^1 isometric embedding $\mathbf{u} : M \rightarrow \mathbb{R}^m$ with $|\mathbf{u} - \bar{\mathbf{u}}|_{C^0} < \epsilon$.

This result is surprising for two reasons. First, the condition (*) is a set of $n(n+1)/2$ equations in m unknowns. A reasonable guess would be that the system is solvable, at least locally, when $m \geq n(n+1)/2$ and this indeed was a classical conjecture of Schläfli (1871). However, for $n \geq 3$ and $m = n + 1$, the system (*) is overdetermined! It is not obvious that there should be any solutions at all, but the Nash-Kuiper Theorem shows that there exists a huge (C^0 -dense) set of solutions in C^1 . Moreover, for $n = 2$ one can compare with classical rigidity results of Herglotz and Cohn-Vossen for the so-called Weyl problem: if (S, g) is a compact Riemannian surface with positive Gauss curvature and if $\mathbf{u} \in C^2$ is an isometric embedding of S into \mathbb{R}^3 , then \mathbf{u} is uniquely determined up to a rigid motion! Thus it is clear that isometric embeddings have very different qualitative behavior at low and high regularity (i.e. C^1 versus C^2). This type of wild non-uniqueness at low regularity is a central aspect of the h-principle introduced by mathematician Mikhail Gromov:

M. Gromov, “A topological technique for the construction of solutions of differential equations and inequalities,” Intern.Congr.Math.(Nice 1970) **2** 221-225 (1971)

M. Gromov, *Partial Differential Relations*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 9 (Springer-Verlag, Berlin, 1986).

with the isometric embedding problem as a primary example. We shall not discuss here the details of Nash’s proof of his remarkable result, but just remark that his construction of the isometry \mathbf{u} was in a series of stages, by adding at each stage a new small, high-frequency perturbation. A brief popular account is here:

<http://www.abelprize.no/c63466/binfil/download.php?tid=63580>

written on the occasion of the award to Nash of the 2015 Abel Prize in mathematics.

The fundamental contribution of De Lellis and Székelyhidi Jr was to realize that there is a very deep mathematical analogy between the problem of isometrically embedding a smooth manifold by a map of low regularity and the problem of solving the Cauchy initial-value problem for incompressible Euler equations by a velocity field of low regularity and that Nash’s method of construction can be carried over to the latter. The analog of a “short mapping” for the Euler system is what De Lellis and Székelyhidi Jr call a smooth subsolution, i.e. a smooth triple $(\bar{\mathbf{v}}, \bar{p}, \boldsymbol{\tau})$ with $\boldsymbol{\tau}$ a symmetric, positive-definite tensor such that

$$\partial_t \bar{\mathbf{v}} + \nabla \cdot (\bar{\mathbf{v}} \bar{\mathbf{v}} + \boldsymbol{\tau}) = -\nabla \bar{p}, \quad \nabla \cdot \bar{\mathbf{v}} = 0.$$

This exactly corresponds to the incompressible Euler equations with the addition of a positive-definite “turbulent stress” tensor $\boldsymbol{\tau}$! The fundamental theorem of Buckmaster et al. (2018) is then the following:

Theorem: Let $(\bar{\mathbf{v}}, \bar{p}, \boldsymbol{\tau})$ be any smooth, strict subsolution of the Euler equations on $\mathbb{T}^3 \times [0, T]$ and let $h < 1/3$. Then there exists a sequence (\mathbf{v}_k, p_k) of weak Euler solutions such that $\mathbf{v}_k \in C^h(\mathbb{T}^3 \times [0, T])$ satisfy, as $k \rightarrow \infty$,

$$\int_{\mathbb{T}^3} d^3x f \mathbf{v}_k \rightarrow \int_{\mathbb{T}^3} d^3x f \bar{\mathbf{v}}, \quad \int_{\mathbb{T}^3} d^3x f \mathbf{v}_k \mathbf{v}_k \rightarrow \int_{\mathbb{T}^3} d^3x f (\bar{\mathbf{v}} \bar{\mathbf{v}} + \boldsymbol{\tau})$$

for all $f \in L^1(\mathbb{T}^3)$ uniformly in time, and furthermore for all $t \in [0, T]$ and all k

$$\int_{\mathbb{T}^3} d^3x \frac{1}{2} |\mathbf{v}_k|^2 = \int_{\mathbb{T}^3} d^3x \frac{1}{2} (|\bar{\mathbf{v}}|^2 + \text{Tr } \boldsymbol{\tau}).$$

This result is an h -principle for weak Euler solutions with Hölder regularity $h < 1/3$. Clearly, there is a huge number of subsolutions, since one may add any positive definite tensor $\boldsymbol{\tau}$ whatsoever. An immediate consequence is therefore

Corollary: Let $e : [0, T] \rightarrow \mathbb{R}^+$ be any strictly positive, smooth function. Then for any $0 < h < 1/3$ there exists a weak Euler solution $\mathbf{v} \in C^h(\mathbb{T}^3 \times [0, T])$ such that

$$\int_{\mathbb{T}^3} d^3x \frac{1}{2} |\mathbf{v}|^2 = e(t).$$

In particular, one may take $e(t)$ to be any function strictly decreasing in time and then the Euler solution of the Corollary (globally) dissipates kinetic energy.

The same “convex integration methods” have also implications for non-uniqueness of the Cauchy initial-value problem, as discussed already by C. De Lellis and L. Székelyhidi Jr (2010). Such results have not yet (to our knowledge) been obtained for any Hölder exponent $h < 1/3$, but have been proved for somewhat lower Hölder regularity. Some recent papers here are

S. Daneri and L. Székelyhidi Jr, “Non-uniqueness and h-principle for Hölder-continuous weak solutions of the Euler equations,” *Arch. Rational Mech. Anal.* **224** 471–514 (2017)

P. Isett, “Nonuniqueness and existence of continuous, globally dissipative Euler flows,” arXiv:1710.11186v2.

We may cite here Theorem 6 from the latter paper:

Theorem: For any $h < 1/15$, the set of divergence-free initial data $\mathbf{v}_0 \in C(\mathbb{T}^3)$ that admit infinitely many incompressible Euler solutions of class C^h that obey local energy conservation

$$\partial_t \left(\frac{1}{2} |\mathbf{v}|^2 \right) + \nabla \cdot \left[\left(\frac{1}{2} |\mathbf{v}|^2 + p \right) \mathbf{v} \right] = 0$$

on a common interval of time containing 0 is C^0 dense.

This situation must be contrasted with that for somewhat smoother initial data, for example $\mathbf{v}_0 \in C^h$ with $h > 1$, where unique solutions of the incompressible Euler equations exist (at least locally in time) that satisfy local kinetic energy conservation and possess the same regularity as the initial data. The non-uniqueness of weak solutions to the Cauchy problem with lower regularity is a “Nash non-rigidity” phenomenon, implying an essential unpredictability.

The constructions of such weak Euler solutions follow a strategy similar to that of Nash, by a sequence of stages $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots$. At stage n one has, after coarse-graining, a subsolution

$$\partial_t \mathbf{v}_n + \nabla \cdot (\mathbf{v}_n \mathbf{v}_n + \boldsymbol{\tau}_n) = -\nabla p_n$$

which is supported on wavenumbers $< 2^n$. By adding a small-scale carefully chosen perturbation one can (together with other operations, such as evolving under smooth Euler dynamics locally in time and “gluing” the different time-segments) succeed to cancel a large part of the stress $\boldsymbol{\tau}_n$ so that, in the limit, $\boldsymbol{\tau}_n \rightarrow \mathbf{0}$ weakly and one obtains a weak limit \mathbf{v} which is a distributional Euler solution. This is therefore a kind of “inverse renormalization group” procedure.

These Euler solutions, one must stress, are not obtained in the physically relevant manner by taking zero-viscosity limits of smooth solutions of the incompressible Navier-Stokes equations. One of the outstanding issues is to show that similar weak Euler solutions are obtained as viscosity tends to 0, at least along suitably selected subsequences of viscosities. If such inviscid-limit solutions do exist, then another open problem is to formulate suitable “admissibility/selection conditions” that such limits must possess and that uniquely characterize those solutions (or classes of such solutions). Some relatively simple proposals are ruled out by the previous results. For example, the local dissipation condition

$$\partial_t \left(\frac{1}{2} u^2 \right) + \partial_x \left(\frac{1}{2} u^3 \right) \leq 0$$

selects a unique weak solution of inviscid Burgers equation, under very modest regularity assumptions. This is not true (even with $= 0!$) for incompressible Euler solutions in C^h , $h < 1/15$.