(C) The Kolmogorov 4/5 - law

We have focussed so far on the absolute structure functions, that were used to obtain bounds on the energy flux Π_ℓ. However, there are other types of structure functions of interest, some of them more directly related to energy flux and which, in fact, provide alternate definitions of it. Rather than define "large-scale energy" by

$$\bar{e}_\ell(\mathbf{x},t) = \tfrac{1}{2}|\bar{\mathbf{u}}_\ell(\mathbf{x},t)|^2$$

one can instead make an alternate definition by filtering just one factor

$$\begin{aligned} e_\ell(\mathbf{x},t) &= \frac{1}{2}\mathbf{u}(\mathbf{x},t)\cdot\bar{\mathbf{u}}_\ell(\mathbf{x},t) \\ &= \int d^d\mathbf{r}\ G_\ell(\mathbf{r})\frac{1}{2}\mathbf{u}(\mathbf{x},t)\cdot\mathbf{u}(\mathbf{x}+\mathbf{r},t) \end{aligned}$$

where

$$e_{\mathbf{r}}(\mathbf{x},t) = \tfrac{1}{2}\mathbf{u}(\mathbf{x},t)\cdot\mathbf{u}(\mathbf{x}+\mathbf{r},t)$$

is the so-called <u>point-split</u> kinetic energy density. The quantities $e_\ell(\mathbf{x},t)$ and $e_{\mathbf{r}}(\mathbf{x},t)$ are well-defined whenever $\mathbf{u}$ has finite mean energy:

$$\frac{1}{T}\int_0^T dt\int_V d^d\mathbf{x}\ \frac{1}{2}u^2(\mathbf{x},t) < +\infty.$$

We now derive a balance equation for the point-split kinetic energy density of a Navier-Stokes solution, as follows

$$\boxed{\begin{aligned} \partial_t(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}') &+ \boldsymbol{\nabla}\cdot\left[(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}')\mathbf{u} + \frac{1}{2}(p\mathbf{u}'+p'\mathbf{u}) + \frac{1}{4}|\mathbf{u}'|^2\delta\mathbf{u} - \nu\boldsymbol{\nabla}(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}')\right] \\ &= \frac{1}{4}\boldsymbol{\nabla}_{\mathbf{r}}\cdot[\delta\mathbf{u}\,|\delta\mathbf{u}|^2] - \nu\boldsymbol{\nabla}\mathbf{u}:\boldsymbol{\nabla}\mathbf{u}' + \frac{1}{2}(\mathbf{f}\cdot\mathbf{u}'+\mathbf{f}'\cdot\mathbf{u}) \end{aligned}}$$

with the notations

$$\begin{aligned} \mathbf{u} &= \mathbf{u}(\mathbf{x},t),\ \ p = p(\mathbf{x},t) \\ \mathbf{u}' &= \mathbf{u}(\mathbf{x}+\mathbf{r},t),\ \ p' = p(\mathbf{x}+\mathbf{r},t) \\ \delta\mathbf{u} &= \mathbf{u}'-\mathbf{u} = \mathbf{u}(\mathbf{x}+\mathbf{r},t)-\mathbf{u}(\mathbf{x},t) \end{aligned}$$

Remarks:

#1. This identity was first derived by L. Onsager in the 1940's in a space-integrated form. The result was communicated to C. C. Lin in a letter in 1945, but never formally published by Onsager. The space-local form was derived by J. Duchon & R. Robert, Nonlinearity **13** 249-255(2000) in a smoothed version, discussed a bit later.

#2. The relation is analogous to the energy balance equation that we derived in the filtering approach:

$$\partial_t(\tfrac{1}{2}|\bar{\mathbf{u}}_\ell|^2) + \boldsymbol{\nabla}\cdot\left[\tfrac{1}{2}|\bar{\mathbf{u}}_\ell|^2\bar{\mathbf{u}}_\ell + \bar{p}_\ell\bar{\mathbf{u}}_\ell - \boldsymbol{\tau}_\ell\cdot\bar{\mathbf{u}}_\ell - \nu\boldsymbol{\nabla}(\tfrac{1}{2}|\bar{\mathbf{u}}_\ell|^2)\right] = \boldsymbol{\nabla}\bar{\mathbf{u}}_\ell : \boldsymbol{\tau}_\ell - \nu|\boldsymbol{\nabla}\bar{\mathbf{u}}_\ell|^2 + \bar{\mathbf{f}}_\ell\cdot\bar{\mathbf{u}}_\ell$$

Proof of the identity: Take

$$\begin{aligned} \partial_t\mathbf{u} + \boldsymbol{\nabla}\cdot(\mathbf{u}\mathbf{u}) - \nu\triangle\mathbf{u} + \boldsymbol{\nabla}p &= \mathbf{f},\ \ \boldsymbol{\nabla}\cdot\mathbf{u} = 0 \\ \partial_t\mathbf{u}' + \boldsymbol{\nabla}\cdot(\mathbf{u}'\mathbf{u}') - \nu\triangle\mathbf{u}' + \boldsymbol{\nabla}p' &= \mathbf{f}',\ \ \boldsymbol{\nabla}\cdot\mathbf{u}' = 0 \end{aligned}$$

Dot the first by $\mathbf{u}'$ and the second by $\mathbf{u}$ and add together, to obtain

$$\begin{aligned} \partial_t(\mathbf{u}\cdot\mathbf{u}') &+ \mathbf{u}\cdot[\boldsymbol{\nabla}\cdot(\mathbf{u}'\mathbf{u}') - \nu\triangle\mathbf{u}'] + \mathbf{u}'\cdot[\boldsymbol{\nabla}\cdot(\mathbf{u}\mathbf{u}) - \nu\triangle\mathbf{u}] \\ &+ \boldsymbol{\nabla}\cdot(p'\mathbf{u} + p\mathbf{u}') = \mathbf{f}'\cdot\mathbf{u} + \mathbf{f}\cdot\mathbf{u}' \end{aligned}$$

Now the viscous term is reorganized as

$$\begin{aligned} \mathbf{u}\cdot\triangle\mathbf{u}' + \mathbf{u}'\cdot\triangle\mathbf{u} &= \boldsymbol{\nabla}\cdot[u_i\boldsymbol{\nabla}u_i' + u_i'\boldsymbol{\nabla}u_i] - 2\boldsymbol{\nabla}u_i\cdot\boldsymbol{\nabla}u_i' \\ &= \boldsymbol{\nabla}\cdot[\boldsymbol{\nabla}(\mathbf{u}\cdot\mathbf{u}')] - 2\boldsymbol{\nabla}\mathbf{u} : \boldsymbol{\nabla}\mathbf{u}' \end{aligned}$$

Lastly we discuss the crucial nonlinear term. Note first that

$$\begin{aligned} \mathbf{u}\cdot[\boldsymbol{\nabla}\cdot(\mathbf{u}'\mathbf{u}')] + \mathbf{u}'\cdot[\boldsymbol{\nabla}\cdot(\mathbf{u}\mathbf{u})] &= u_i\partial_j(u_i'u_j') + u_i'\partial_j(u_iu_j) \\ &= u_i\partial_j(u_i'u_j') + \partial_j(u_i'u_iu_j) - u_iu_j(\partial_ju_i') \\ &= \triangle + \boldsymbol{\nabla}\cdot[(\mathbf{u}\cdot\mathbf{u}')\mathbf{u}] \end{aligned}$$

with

$$\triangle \equiv u_i\partial_j(u_i'u_j') - u_iu_j(\partial_ju_i'))$$

.

By incompressibility,

$$\begin{aligned}
\triangle &= u_i u_j'(\partial_j u_i') - u_i u_j(\partial_j u_i') \\
&= u_i(u_j' - u_j)(\partial_j u_i') \\
&= u_i \delta u_j(\partial_j u_i')
\end{aligned}$$

Also by incompressibility $\boldsymbol{\nabla}_{\mathbf{r}} \cdot (\delta\mathbf{u}) = \boldsymbol{\nabla}_{\mathbf{r}} \cdot \mathbf{u}' = 0$, so that

$$\begin{aligned}
\boldsymbol{\nabla}_{\mathbf{r}} \cdot [\delta\mathbf{u}\,|\delta\mathbf{u}|^2] &= (\delta\mathbf{u} \cdot \boldsymbol{\nabla}')|\delta\mathbf{u}|^2 \\
&= 2(\delta u_j \partial_j) u_i' \cdot (u_i' - u_i) \\
&= 2\delta u_j \cdot u_i' \partial_j u_i' - 2u_i \delta u_j(\partial_j u_i') \\
&= \delta\mathbf{u} \cdot \boldsymbol{\nabla}(|\mathbf{u}'|^2) - 2\triangle \\
&= \boldsymbol{\nabla} \cdot [|\mathbf{u}'|^2 \delta\mathbf{u}] - 2\triangle
\end{aligned}$$

Thus,

$$\begin{aligned}
\partial_t(\mathbf{u}\cdot\mathbf{u}') &+ \boldsymbol{\nabla} \cdot \left[(\mathbf{u}\cdot\mathbf{u}')\mathbf{u} + (p\mathbf{u}' + p'\mathbf{u}) + \frac{1}{2}|\mathbf{u}'|^2\delta\mathbf{u} - \nu\boldsymbol{\nabla}(\mathbf{u}\cdot\mathbf{u}')\right] \\
&- \frac{1}{2}\boldsymbol{\nabla}_{\mathbf{r}} \cdot [\delta\mathbf{u}\,|\delta\mathbf{u}|^2] + 2\nu\boldsymbol{\nabla}\mathbf{u} : \boldsymbol{\nabla}\mathbf{u}' = (\mathbf{f}\cdot\mathbf{u}' + \mathbf{f}'\cdot\mathbf{u})
\end{aligned}$$

QED!

Multiplying the point-split identity through by $G_\ell(\mathbf{r})$ and integrating over $\mathbf{r}$ gives a corresponding balance equation for the regularized energy density $\frac{1}{2}\mathbf{u}\cdot\bar{\mathbf{u}}_\ell$:

$$\begin{aligned}
\partial_t(\frac{1}{2}\mathbf{u}\cdot\bar{\mathbf{u}}_\ell) &+ \boldsymbol{\nabla} \cdot \left[(\frac{1}{2}\mathbf{u}\cdot\bar{\mathbf{u}}_\ell)\mathbf{u} + \frac{1}{2}(p\bar{\mathbf{u}}_\ell + \bar{p}_\ell\mathbf{u}) + \frac{1}{4}\overline{(|\mathbf{u}|^2\mathbf{u})_\ell} - \frac{1}{4}\overline{(|\mathbf{u}|^2)_\ell}\mathbf{u} - \nu\boldsymbol{\nabla}(\frac{1}{2}\mathbf{u}\cdot\bar{\mathbf{u}}_\ell)\right] \\
&= -\frac{1}{4\ell}\int d^d\mathbf{r}\,(\boldsymbol{\nabla}G)_\ell(\mathbf{r})\cdot\delta\mathbf{u}(\mathbf{r})\,|\delta\mathbf{u}(\mathbf{r})|^2 - \nu\boldsymbol{\nabla}\mathbf{u} : \boldsymbol{\nabla}\bar{\mathbf{u}}_\ell + \frac{1}{2}(\mathbf{f}\cdot\bar{\mathbf{u}}_\ell + \bar{\mathbf{f}}_\ell\cdot\mathbf{u})
\end{aligned}$$

The above equation can be shown to be valid even for singular Leray solutions of INS, if the space-time derivatives are interpreted in the sense of distributions. (See Appendix.)

We now consider the limit of vanishing viscosity $\nu \to 0$. If the Navier-Stokes solution $\mathbf{u}^\nu \to \mathbf{u}$ as $\nu \to 0$ in the space-time L^2-sense, i.e.

$$\int dt \int d^d\mathbf{x}\ |\mathbf{u}^\nu(\mathbf{x},t) - \mathbf{u}(\mathbf{x},t)|^2 \to 0, \text{ as } \nu \to 0$$

then it is not hard to show that the limiting velocity $\mathbf{u}$ is a solution of the incompressible Euler equations

$$\partial_t \mathbf{u} + \boldsymbol{\nabla}\cdot(\mathbf{u}\mathbf{u}) = -\boldsymbol{\nabla}p + \mathbf{f},\ \boldsymbol{\nabla}\cdot\mathbf{u} = 0$$

in the sense of space-time distributions. If it is furthermore true that $\int dt \int d^d\mathbf{x}\ |\mathbf{u}(\mathbf{x},t)|^3 < +\infty$, then the point-split balance holds also for the Euler solution in the distribution sense:

$$\begin{aligned}\partial_t(\frac{1}{2}\mathbf{u}\cdot\bar{\mathbf{u}}_\ell) + \boldsymbol{\nabla}\cdot\left[(\frac{1}{2}\mathbf{u}\cdot\bar{\mathbf{u}}_\ell)\mathbf{u} + \frac{1}{2}(p\bar{\mathbf{u}}_\ell + \bar{p}_\ell\mathbf{u}) + \frac{1}{4}\overline{(|\mathbf{u}|^2\mathbf{u})_\ell} - \frac{1}{4}\overline{(|\mathbf{u}|^2)_\ell}\mathbf{u}\right] \\ = -\frac{1}{4\ell}\int d^d\mathbf{r}\,(\boldsymbol{\nabla}G)_\ell(\mathbf{r})\cdot\delta\mathbf{u}(\mathbf{r})\,|\delta\mathbf{u}(\mathbf{r})|^2 \qquad (\star)\end{aligned}$$

For simplicity, we consider the case with no external force, $\mathbf{f} = 0$. We now consider the limit $\ell \to 0$. Under the same basic assumption, that $\int dt \int d^d\mathbf{x}|\mathbf{u}(\mathbf{x},t)|^3 < +\infty$, it is not hard to show that the LHS of equation $(\star)$

$$\begin{aligned}\partial_t(\frac{1}{2}\mathbf{u}\cdot\bar{\mathbf{u}}_\ell) \quad &+ \quad \boldsymbol{\nabla}\cdot\left[(\frac{1}{2}\mathbf{u}\cdot\bar{\mathbf{u}}_\ell)\mathbf{u} + \frac{1}{2}(p\bar{\mathbf{u}}_\ell + \bar{p}_\ell\mathbf{u}) + \frac{1}{4}\overline{(|\mathbf{u}|^2\mathbf{u})_\ell} - \frac{1}{4}\overline{(|\mathbf{u}|^2)_\ell}\mathbf{u}\right] \\ &\longrightarrow \quad \partial_t(\frac{1}{2}|\mathbf{u}|^2) + \boldsymbol{\nabla}\cdot\left[(\frac{1}{2}|\mathbf{u}|^2 + p)\mathbf{u}\right] \equiv -D(\mathbf{u}), \quad \text{as } \ell \to 0\end{aligned}$$

in the sense of distributions. Just to consider one typical term,

$$\begin{aligned}&\left|\int dt \int d^d\mathbf{x}\ \boldsymbol{\nabla}\varphi(\mathbf{x},t)\cdot(\frac{1}{2}\mathbf{u}(\mathbf{x},t)\cdot\bar{\mathbf{u}}_\ell(\mathbf{x},t))\mathbf{u}(\mathbf{x},t) - \int dt \int d^d\mathbf{x}\ \boldsymbol{\nabla}\varphi(\mathbf{x},t)\cdot\mathbf{u}(\mathbf{x},t)\frac{1}{2}|\mathbf{u}(\mathbf{x},t)|^2\right| \\ \leq\ &\sup|\boldsymbol{\nabla}\varphi|\cdot||(\frac{1}{2}\mathbf{u}\cdot\bar{\mathbf{u}}_\ell)\mathbf{u} - (\frac{1}{2}|\mathbf{u}|^2)\mathbf{u}||_{L^1_{\text{spacetime}}}\end{aligned}$$

Using $\frac{1}{2}(\mathbf{u}\cdot\bar{\mathbf{u}}_\ell)\mathbf{u} - \frac{1}{2}|\mathbf{u}|^2\mathbf{u} = \frac{1}{2}[\mathbf{u}\cdot(\bar{\mathbf{u}}_\ell - \mathbf{u})]\mathbf{u}$ and the Hölder inequality

$$||[\mathbf{u}\cdot(\bar{\mathbf{u}}_\ell - \mathbf{u})]\mathbf{u}||_{L^1} \leq ||\mathbf{u}||^2_{L^3}||\bar{\mathbf{u}}_\ell - \mathbf{u}||_{L^3}$$

One can then show that the upper bound $\to 0$ as $\ell \to 0$. This implies that

$$\boldsymbol{\nabla}\cdot\left[(\tfrac{1}{2}\mathbf{u}\cdot\bar{\mathbf{u}}_\ell)\mathbf{u}\right] \longrightarrow \boldsymbol{\nabla}\cdot\left[\tfrac{1}{2}|\mathbf{u}|^2\mathbf{u}\right]$$

in the sense of distributions. The other terms are treated in a similar fashion. But since the LHS of equation $(\star)$ converges to $-D(\mathbf{u})$ in the sense of the distributions, so does the RHS! That is,

$$\boxed{D(\mathbf{u}) = \lim_{\ell\to 0} \frac{1}{4\ell} \int d^d\mathbf{r}\ (\boldsymbol{\nabla} G)_\ell(\mathbf{r}) \cdot \delta\mathbf{u}(\mathbf{r}) |\delta\mathbf{u}(\mathbf{r})|^2 \qquad (1)}$$

in the sense of distributions. To summarize, we obtain the above formula for the anomalous dissipation $D(\mathbf{u})$ that appears in the energy balance relation

$$\boxed{\partial_t(\frac{1}{2}|\mathbf{u}|^2) + \boldsymbol{\nabla} \cdot \left[(\frac{1}{2}|\mathbf{u}|^2 + p)\mathbf{u}\right] = -D(\mathbf{u}). \qquad (2)}$$

for the singular Euler solution $\mathbf{u}(\mathbf{x}, t)$. This result is quite interesting in its own right and not just as a step in the proof of the Kolmogorov 4/5-law. It is a precise mathematical formulation of Onsager's idea that Euler solutions which arises in the zero-viscosity limit of turbulent flow may not conserve energy. We could derive the same balance equation (2) by starting from the balance equation for $\frac{1}{2}|\bar{\mathbf{u}}_\ell|^2$ and taking the limit $\ell \to 0$. This would give us another valid expression

$$D(\mathbf{u}) = \lim_{\ell\to 0} \Pi_\ell \quad \text{(in the distribution sense)}$$

for the anomalous dissipation $D(\mathbf{u})$. In fact, the RHS of equation $(\star)$

$$D_\ell(\mathbf{u}) = \frac{1}{4\ell} \int d^d\mathbf{r}\ (\boldsymbol{\nabla} G)_\ell(\mathbf{r}) \cdot \delta\mathbf{u}(\mathbf{r}) |\delta\mathbf{u}(\mathbf{r})|^2 \qquad (3)$$

is another way of measuring energy flux to small scales, alternative to Π_ℓ. We can get from equation (1) for $D(\mathbf{u})$ Onsager's bound on energy flux to small scales. For example, if $\mathbf{u}$ has Hölder exponent h, then it follows from equation (3) that

$$D_\ell(\mathbf{u}) = O(\ell^{3h-1})$$

the same bound derived earlier for Π_ℓ. These bounds imply the assertion of Onsager in his 1949 paper that Euler solutions must conserve energy if the velocity has Hölder exponent $h > 1/3$. Using L_p norms, one can easily show also that energy is conserved if $\sigma_p > \frac{1}{3}$ (equivalently, $\zeta_p > p/3$) for $p \geq 3$. Under any of these regularity assumptions, $D(\mathbf{u}) \equiv 0$!

Let us now return to our derivation of the 4/5-law, by obtaining a simplified expression for $D(\mathbf{u})$ for the case of a spherically symmetric filter kernel G that depends upon only the magnitude $r = |\mathbf{r}|$:

$$G(\mathbf{r}) = G(r)$$

so that

$$\boldsymbol{\nabla} G(\mathbf{r}) = \hat{\mathbf{r}} G'(r).$$

In that case, one can go to spherical coordinates in d-dimensions

$$D_\ell(\mathbf{u}) = \tfrac{1}{4\ell}\int_0^\infty r^{d-1}dr\ \int_{S^{d-1}} d\omega(\hat{\mathbf{r}})\hat{\mathbf{r}}\cdot\delta\mathbf{u}(\mathbf{r})|\delta\mathbf{u}(\mathbf{r})|^2(G')_\ell(r)$$

where S^{d-1} is the unit sphere in d-dimensions and $d\omega$ is the measure on solid angles. Now introduce

$$\delta u_L(\mathbf{r}) = \hat{\mathbf{r}}\cdot\delta\mathbf{u}(\mathbf{r}) = \underline{\text{longitudinal velocity increment}}$$

and

$$\begin{aligned}\langle \delta u_L(\mathbf{r})|\delta\mathbf{u}(\mathbf{r})|^2\rangle_{ang} &= \frac{1}{\Omega_{d-1}}\int_{S^{d-1}} d\omega(\hat{\mathbf{r}})\delta u_L(\mathbf{r})|\delta\mathbf{u}(\mathbf{r})|^2 \\ &= \text{angular average of } \ \delta u_L|\delta\mathbf{u}|^2\end{aligned}$$

where Ω_{d-1} is the $(d-1)$ -dimension volume of S^{d-1}. We thus find that

$$\begin{aligned}D_\ell(\mathbf{u}) &= \frac{1}{4\ell}\Omega_{d-1}\int_0^\infty r^{d-1}dr\ (G')_\ell(r)\langle\delta u_L(\mathbf{r})|\delta\mathbf{u}(\mathbf{r})|^2\rangle_{ang} \\ &= \Omega_{d-1}\int_0^\infty \rho^d d\rho\ G'(\rho)\ \left.\frac{\langle\delta u_L(\mathbf{r})|\delta\mathbf{u}(\mathbf{r})|^2\rangle_{ang}}{4r}\right|_{\mathbf{r}=\ell\boldsymbol{\rho}}\end{aligned}$$

where $\boldsymbol{\rho} = \mathbf{r}/\ell$. We know that the limit of the LHS exists as $\ell \to 0$ in the sense of distributions and gives $D(\mathbf{u})$. Taking the limit on the RHS, we see that

$$\tfrac{\langle\delta u_L(\mathbf{r})|\delta\mathbf{u}(\mathbf{r})|^2\rangle_{ang}}{4r} \longrightarrow D^*(\mathbf{u})\ \text{, as}\quad r\to 0$$

with

$$\begin{aligned}D(\mathbf{u}) &= D^*(\mathbf{u})\cdot\Omega_{d-1}\int_0^\infty \rho^d d\rho G'(\rho) \\ &= D^*(\mathbf{u})\cdot(-d\cdot\Omega_{d-1}\int_0^\infty \rho^{d-1}d\rho G(\rho))\quad\text{by integration by parts} \\ &= -d\cdot D^*(\mathbf{u})\ \ \text{since}\ \ \Omega_{d-1}\int_0^\infty \rho^{d-1}d\rho G(\rho) = 1\end{aligned}$$

We conclude finally that

$$\boxed{\lim_{r\to 0} \frac{\langle \delta u_L(\mathbf{r})|\delta \mathbf{u}(\mathbf{r})|^2\rangle_{ang}}{r} = -\frac{4}{d} D(\mathbf{u})}$$

The results in the above form were given in the paper

J. Duchon & R. Robert, "Inertial energy dissipation for weak solution of incompressible Euler and Navier-Stokes equations," Nonlinearity, **13** 249-255(2000).

It is possible, by an elaboration of these arguments, to derive expressions for $D(\mathbf{u})$ that involve only $\delta u_L(\mathbf{r})$, or mixed expressions that involve $\delta u_L(\mathbf{r})$ and the transverse velocity increment

$$\delta \mathbf{u}_T(\mathbf{r}) = \delta \mathbf{u}(\mathbf{r}) - \delta u_L(\mathbf{r})\hat{\mathbf{r}}$$

which satisfies $\hat{\mathbf{r}} \cdot \delta \mathbf{u}_T(\mathbf{r}) = 0$, or

$$\delta u_T^2(\mathbf{r}) = |\delta \mathbf{u}_T(\mathbf{r})|^2/(d-1),$$

the magnitude of the transverse velocity increment per component. These are, in d-dimensions,

$$\boxed{\begin{array}{rcl} \lim_{r\to 0} \frac{\langle \delta u_L^3(\mathbf{r})\rangle_{ang}}{r} & = & -\frac{12}{d(d+2)} D(\mathbf{u}) \\ \lim_{r\to 0} \frac{\langle \delta u_L(\mathbf{r})\delta u_T^2(\mathbf{r})\rangle_{ang}}{r} & = & -\frac{4}{d(d+2)} D(\mathbf{u}) \end{array}}$$

For the derivation, see G. L. Eyink, Nonlinearity **16**, 137-145(2003). The idea is to look at separate equations for point-split energy densities $\mathbf{u}(\mathbf{x},t) \cdot \mathbf{u}_L(\mathbf{x}+\mathbf{r},t)$, $\mathbf{u}(\mathbf{x},t) \cdot \mathbf{u}_T(\mathbf{x}+\mathbf{r},t)$.

Example: Burgers Equation

The above discussion has been a bit abstract, so that it is useful to consider a concrete example. All the previous results have exact analogies for singular/distributional solutions of the inviscid Burgers equation, which can be shown to satisfy the energy balance equation

$$\partial_t(\tfrac{1}{2}u^2) + \partial_x(\tfrac{1}{3}u^3) = -D(u)$$

with

$$D(u) = \lim_{\ell\to 0} \frac{1}{12\ell}\int_{-\infty}^{+\infty} dr (G')_\ell(r)\delta u_L^3(r)$$

in the sense of distributions. Alternately,

$$\lim_{r\to 0} \tfrac{\langle \delta u_L^3(r)\rangle_{ang}}{|r|} = -12D(u)$$

where $\langle \delta u_L^3(r)\rangle_{ang} = \frac{1}{2}[\delta u^3(+|r|) - \delta u^3(-|r|)]$. For the Khokhlov sawtooth solution in the limit $\nu \to 0$ it is straightforward to calculate explicitly that, with $r > 0$

$$\begin{aligned}\langle \delta u_L^3(r)\rangle_{ang} &= \frac{1}{2}[(r/t - \triangle u)^3 - (-r/t)^3]\chi_{[-r,0]}(x) \\ &\quad + \frac{1}{2}[(r/t)^3 - (-r/t + \triangle u)^3]\chi_{[0,r]}(x)\end{aligned}$$

so that

$$\tfrac{\langle \delta u_L^3(r)\rangle_{ang}}{r} \longrightarrow -[\tfrac{1}{2}(\triangle u)^3 + \tfrac{1}{2}(\triangle u)^3]\delta(x) = -(\triangle u)^3\delta(x) \ \text{, as} \quad r \to 0.$$

Notice that this is equal to $-12\varepsilon(x)$, where $\varepsilon(x) = \lim_{r\to 0}\nu(\partial_x u^\nu)^2 = \frac{1}{12}(\Delta u)^3\delta(x)$ is the distributional limit of the viscous dissipation in $u^\nu(x,t)$ as $\nu \to 0$.

A similar result can be obtained for the $\nu \to 0$ limit of Leray solutions of the Navier-Stokes equation. These satisfy a local energy balance of the form

$$\partial_t(\frac{1}{2}|\mathbf{u}^\nu|^2) + \boldsymbol{\nabla}\cdot\left[(\frac{1}{2}|\mathbf{u}^\nu|^2 + p^\nu)\mathbf{u}^\nu - \nu\boldsymbol{\nabla}(\frac{1}{2}|\mathbf{u}^\nu|^2)\right] = -\nu|\boldsymbol{\nabla}\mathbf{u}^\nu|^2 \quad (\text{or, } \leq -\nu|\boldsymbol{\nabla}\mathbf{u}^\nu|^2), \qquad (\star)$$

For simplicity, we shall only consider the case where "=" holds above rather than "≤". (For the general case, see Appendix.) Let us assume that $\mathbf{u}^\nu \to \mathbf{u}$ as $\nu \to 0$ in the L^3-sense in spacetime, i.e.

$$\int dt \int d^dx \ |\mathbf{u}^\nu(\mathbf{x},t) - \mathbf{u}(\mathbf{x},t)|^3 \longrightarrow 0.$$

This is stronger than the L^2-convergence assumed earlier, so that, again, the limiting velocity $\mathbf{u}$ is an Euler solution in distribution sense. Furthermore, it is now possible to check that the LHS of equation $(\star)$ above has the limit

$$\lim_{\nu\to 0}\partial_t(\tfrac{1}{2}|\mathbf{u}^\nu|^2) + \boldsymbol{\nabla}\cdot\left[(\tfrac{1}{2}|\mathbf{u}^\nu|^2 + p^\nu)\mathbf{u}^\nu - \nu\boldsymbol{\nabla}(\tfrac{1}{2}|\mathbf{u}^\nu|^2)\right] = \partial_t(\tfrac{1}{2}|\mathbf{u}|^2) + \boldsymbol{\nabla}\cdot\left[(\tfrac{1}{2}|\mathbf{u}|^2 + p)\mathbf{u}\right]$$

in distribution sense. The argument is very similar to that which we gave earlier for the limit $\ell \to 0$. Furthermore, the limit is exactly the same, i.e. $-D(\mathbf{u})$! Since the limits of the LHS and the RHS of equation ($\star$) must be the same, we obtain

$$\boxed{D(\mathbf{u}) = \lim_{\nu\to 0} \nu|\boldsymbol{\nabla}\mathbf{u}^\nu|^2 = \lim_{\nu\to 0} \varepsilon^\nu \quad \text{(Duchon \& Robert, 2000)}}$$

in the sense of distributions. Notice the RHS of the above expression is non-negative, so that its limit also must be:

$$D(\mathbf{u}) \geq 0$$

More precisely, $D(\mathbf{u})$ is a nonnegative distribution, which satisfies $\int d^d\mathbf{x}\, dt\, \varphi(\mathbf{x},t) D(\mathbf{u})(\mathbf{x},t) \geq 0$ for every nonnegative test function φ (C^∞ with compact support). It is known that every nonnegative distribution is given by a measure, i.e.

$$\int d^d\mathbf{x} \int dt\, \varphi(\mathbf{x},t) D(\mathbf{u})(\mathbf{x},t) = \int\int \mu(d\mathbf{x}, dt)\, \varphi(\mathbf{x},t)$$

This "dissipation measure" has been much studied experimentally and observed to have multi-fractal scaling properties, as we discuss a bit later! If φ is nonnegative and also normalized

$$\int dt \int d^d\mathbf{x}\, \varphi(\mathbf{x},t) = 1,$$

then we can interpret

$$\int dt \int d^d\mathbf{x}\, \varphi(\mathbf{x},t)\, \nu|\boldsymbol{\nabla}\mathbf{u}^\nu(\mathbf{x},t)|^2 \equiv \langle \nu|\boldsymbol{\nabla}\mathbf{u}^\nu|^2\rangle_\varphi$$

as an average in spacetime over the compact support of φ, weighted by φ. The above result then says that

$$\lim_{\nu\to 0}\langle \varepsilon^\nu\rangle_\varphi = \langle D(\mathbf{u})\rangle_\varphi$$

Our earlier results can be stated in a similar fashion, e.g.

$$\lim_{r\to 0}\lim_{\nu\to 0} \frac{\langle [\delta u_L^\nu(\mathbf{r})]^3\rangle_{\varphi,ang}}{r} = -\frac{12}{d(d+2)}\langle D(\mathbf{u})\rangle_\varphi.$$

We thus see that, taking first $\nu \to 0$,

$$\boxed{\langle \delta u_L^3(\mathbf{r})\rangle_{\varphi,ang} \sim -\frac{12}{d(d+2)}\langle\varepsilon\rangle_\varphi r}$$

This is the famous Kolmogorov 4/5-law (since the coefficient $\frac{12}{d(d+2)} = \frac{4}{5}$ for $d = 3$), derived by Kolmogorv in the third of his celebrated 1941 papers on turbulence. The related results

$$\boxed{\begin{aligned} \langle \delta u_L(\mathbf{r}) \delta u_T^2(\mathbf{r}) \rangle_{\varphi,ang} &\sim -\frac{4}{d(d+2)} \langle \varepsilon \rangle_\varphi r \\ \langle \delta u_L(\mathbf{r}) |\delta \mathbf{u}(\mathbf{r})|^2 \rangle_{\varphi,ang} &\sim -\frac{4}{d} \langle \epsilon \rangle_\varphi r \end{aligned}}$$

are called the Kolmogorov 4/15- and 4/3-laws, respectively. These were derived by Kolmogorov in the statistical sense, averaging over an ensemble of solutions assuming statistical homogeneity and isotropy. He employed in his derivation an equation derived earlier for the 2-point velocity correlation $\langle u_i(\mathbf{x}, t) u_j(\mathbf{x} + \mathbf{r}, t) \rangle$ by von Kármán and Howarth (1938), so that this is sometimes called the Kolmogorov-Kármán-Howarth relation. The result presented here (following largely the derivation of Duchon & Robert, 2000) is much stronger, because there is no average over ensembles and no assumption of homogeneity and/or isotropy. It seems to have been Onsager in the 1940's who realized that such relations should hold for individual realizations, without averaging. He derived the formula

$$D_\ell(\mathbf{u}) = \frac{1}{4\ell} \int d^d r \, (\boldsymbol{\nabla} G)_\ell(\mathbf{r}) \cdot \delta \mathbf{u}(\mathbf{r}) |\delta \mathbf{u}(\mathbf{r})|^2$$

and discussed its limit for $\ell \to 0$. In the statistical framework, the corresponding result

$$\boldsymbol{\nabla}_\mathbf{r} \cdot \langle \delta \mathbf{u}(\mathbf{r}) |\delta \mathbf{u}(\mathbf{r})|^2 \rangle \sim -4 \langle \varepsilon \rangle \quad , \text{as} \quad r \to 0$$

was derived by A. S. Monin (1959) and is sometimes called the Kolmogorov-Monin relation. It does not assume isotropy. There is another derivation of the 4/5-law by Nie & Tanveer (1999) without statistical averaging. It uses also spacetime averaging and angle-averaging. It is stronger than the result presented here in that it includes viscous corrections, but it is weaker than the presented local results, since it requires a global spacetime average.

Experiments and Simulations

⋆ K. Sreenivasan et al., PRL, vol.77, 1488-1491 (1996)

These authors present data for the 4/5-law obtained from experimental observations of the centerline of flow through a pipe at bulk Reynolds number $Re = 230,000$, and from a 512^3 DNS of homogeneous forced turbulence at $Re_\lambda = 220(\doteq \sqrt{Re})$. The data for velocity increments were not angle-averaged, so this is also a test of return to isotropy at small scales.

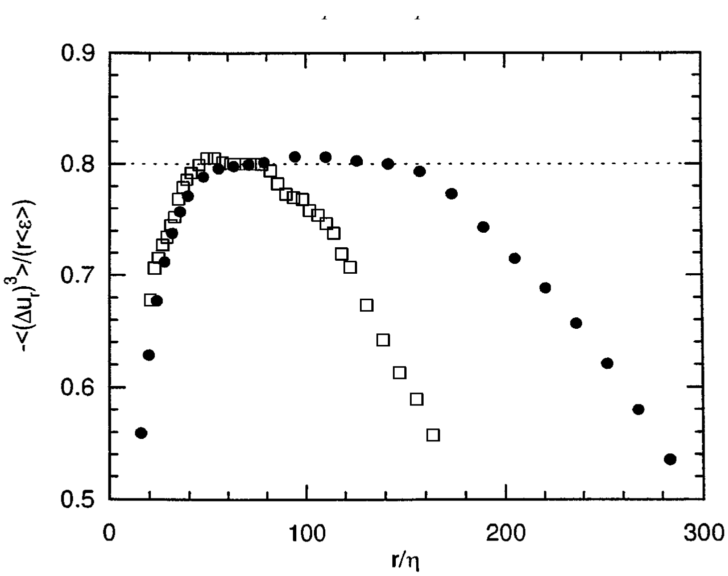

FIG. 2. The quantity $\langle \Delta u_r^3 \rangle / r$ as a function of r. Squares, experiment; circles, simulations; dots indicate Kolmogorov's $\frac{4}{5}$th law. It is believed that the slight bump in the left part of the experimental data is the bottleneck effect [see G. Falkovich, Phys. Fluids **6**, 1411 (1994); D. Lohse and A. Mueller-Groeling, Phys. Rev. Lett. **74**, 1747 (1995)]. While the bottleneck effects discussed in these two papers refer especially to second-order structure functions (or to energy spectrum), a similar effect is likely to exist for the third-order as well. This is typical of most measurements [see, for example, Y. Gagne, Docteur ès-Sciences Physiques Thèse, Université de Grenoble, France (1987)].

⋆ T. Gotoh et al. Phys. Fluids **14**, 1065-1081 (2002)

This paper presents data from numerical simulations up to 1024^3 resolution of homogenous forced turbulence, and Re_λ in the range $38 - 460$. Tests were made of both the 4/5-law and the 4/3-law, again without angle-averaging.

Velocity field statistics in homogeneous 1071

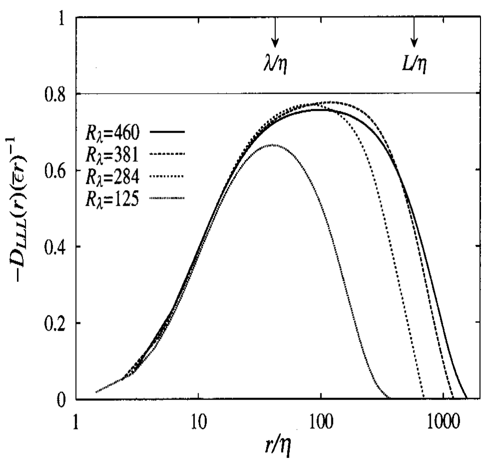

FIG. 12. Kolmogorov's 4/5 law. L/η and λ/η are shown for $R_\lambda = 460$. The maximum values of the curves are 0.665, 0.771, 0.781, and 0.757 for $R_\lambda = 125$, 284, 381, and 460, respectively.

1072 Phys. Fluids, Vol. 14, No. 3, March 2002

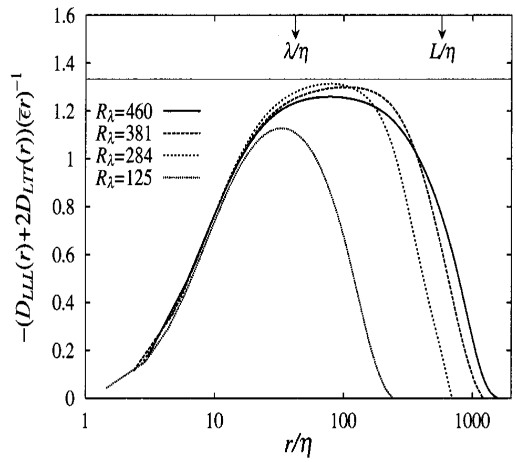

FIG. 14. Kolmogorov's 4/3 law. L/η and λ/η are shown for $R_\lambda = 460$. The maximum values of the curves for the 4/3 law are 0.564, 1.313, 1.297, and 1.259 for $R_\lambda = 125$, 284, 381, and 460, respectively.

⋆ M. Taylor et al. Phys. Rev. E **68**, 026310 (2003)

Another numerical study in a 512^3 DNS of homogenous forced turbulence, with $Re_\lambda \cong 249 - 263$. This paper studies the effects of angle-averaging and obtains results with such averaging comparable to those of Gotoh et al.(2002) at nearly twice the Reynolds number.

TAYLOR, KURIEN, AND EYINK PHYSICAL REVIEW E **68**, 026310 (2003)

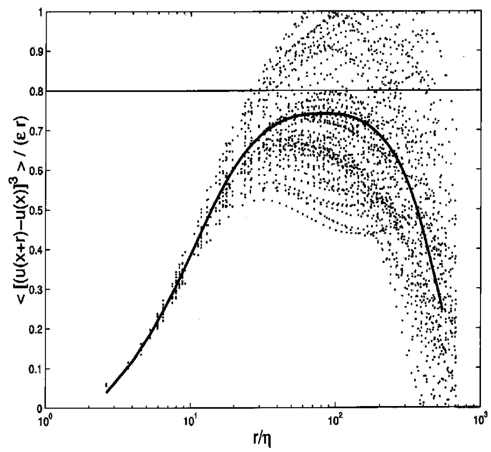

FIG. 6. The nondimensional third-order longitudinal structure function, computed from a single snapshot of the stochastic dataset, vs the nondimensional scale r/η. The dots indicate the values of the structure function computed at various $\ell\mathbf{r}_j$. The thick curve is the angle average. The horizontal line indicates the 4/5 mark.

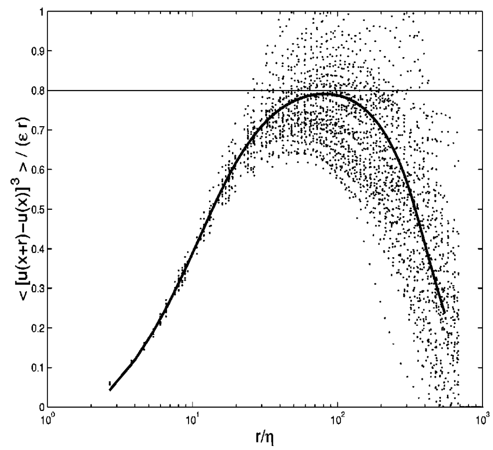

FIG. 7. The nondimensional third-order longitudinal structure function computed from a single snapshot of the deterministic dataset vs the nondimensional scale r/η. The various symbols and lines mean the same as in Fig. 6.

<u>One last remark:</u> The derivation that we have given applies even if $D(\mathbf{u}) \equiv 0$, i.e. vanishes everywhere. For example, this holds in a smooth solution of the Euler equations, for which $\langle \delta u_L^3(\mathbf{r})\rangle_{ang} \sim \langle \delta u_L(\mathbf{r})\delta \mathbf{u}_T^2(\mathbf{r})\rangle \sim O(r^3)$, so that $D_\ell(\mathbf{u}) = O(\ell^2) \to 0$ as $\ell \to 0$. Another example is 2D Euler solutions where, under very general assumptions, $D(\mathbf{u}) \equiv 0$ and there is no energy cascade to small scales. E.g. see Proposition 6 in Duchon & Robert (2000). There is a nontrivial extension of the $\frac{4}{5}$-law to 2D turbulence, but with $\langle \delta u_L^3(\mathbf{r})\rangle_{ang}$ positive, corresponding to inverse energy cascade. E.g. see D. Bernard (1999).

<u>Additional References:</u>

T. von Kármán & L. Howarth, "On the statistical theory of isotropic turbulence," Proc. Roy. Soc. Lond. A **164**, 192-215 (1938).

A. N. Kolmogorov, "Dissipation of energy in locally isotropic turbulence," Dokl. Akad. Nauk. SSR **32**, 16-18 (1941).

A. S. Monin, “Theory of locally isotropic turbulence,” Dokl. Akad. Nauk. SSSR **125** 515-518(1959); see also, A.S. Monin & A. M. Yaglom, Statistical Fluid Mechanics, vol.2 (MIT, 1975), p.403.

Papers on Deterministic Versions of $\frac{4}{5}$-law

Q. Nie & S. Tanveer, “A note on the third-order struncture functions in turbulence,” Proc. R. Soc. A **455**, 1615-1635(1999).

J. Duchon & Robert, “Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes,” Nonlinearity, **13** 249-255(2000)

G. L. Eyink, “Local $\frac{4}{5}$-law and energy dissipation anomaly in turbulence,” Nonlinearity, **16** 137-145(2003).

G. L. Eyink, “Onsager and the theory of hydrodynamic turbulence,” Rev. Mod. Phys. **78** 87-135(2006), Section IV, B.

2D Analogues of the $\frac{4}{5}$-law

D. Bernard, “Three-point velocity correlation functions in two-dimensional forced turbulence,” Phys. Rev. E **60** 6184-6187(1993).

A. M. Polyakov, “The theory of turbulence in two dimensions,” Nucl. Phys. B **396**, 367-385(1993). This paper, in particular, discusses the analogy of the $\frac{4}{5}$-law with conservation-law anomalies in quantum field theories.

APPENDIX

The purpose of this appendix is to discuss more carefully the issue of energy balance/energy dissipation for Leray Solutions of the incompressible Navier-Stokes (INS) equations, which are possibly singular. It was proved by

J. Leray, "Essai sur Le Mouvement d'un fluide visqueux emplissant l'espace," Acta. Math. **63** 193-248 (1934)

that solutions of the INS

$$\partial_t \mathbf{u}^\nu + \boldsymbol{\nabla}\cdot(\mathbf{u}^\nu\mathbf{u}^\nu) = -\boldsymbol{\nabla} p^\nu + \nu\triangle\mathbf{u}^\nu, \quad \boldsymbol{\nabla}\cdot\mathbf{u}^\nu = 0$$

always exist in the sense of spacetime distributions, i.e.

$$\begin{aligned} & \int_{t_0}^T dt \int d^d\mathbf{x}[\partial_t\boldsymbol{\varphi}(\mathbf{x},t)\cdot\mathbf{u}^\nu(\mathbf{x},t) + \boldsymbol{\nabla}\boldsymbol{\varphi}(\mathbf{x},t) : \mathbf{u}^\nu(\mathbf{x},t)\otimes\mathbf{u}^\nu(\mathbf{x},t) \\ + & \quad (\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}(\mathbf{x},t))p^\nu(\mathbf{x},t) - \nu\triangle\boldsymbol{\varphi}(\mathbf{x},t)\cdot\mathbf{u}^\nu(\mathbf{x},t)] = \int d^d\mathbf{x}\;\; \boldsymbol{\varphi}(\mathbf{x},t_0)\cdot\mathbf{u}_0^\nu(\mathbf{x}), \end{aligned}$$

$$\int_{t_0}^T dt \;\int d^d\mathbf{x}\boldsymbol{\nabla}\psi(\mathbf{x},t)\cdot\mathbf{u}^\nu(\mathbf{x},t) = 0$$

for any initial condition $\mathbf{u}_0^\nu$ with finite energy: $||\mathbf{u}_0^\nu||_{L^2(V)} < +\infty$. Here $\boldsymbol{\varphi} = (\varphi_1,\cdots,\varphi_d)$, ψ are C^∞ functions on spacetime with compact support. Note that the above rather abstract-looking formulation is actually very physical! It is equivalent to the following balance equation for the momentum in a bounded region $\Omega \subseteq V$ with smooth boundary $\partial\Omega$ and times $t > t' \geq t_0$

$$\begin{aligned} & \int_\Omega \mathbf{u}^\nu(\mathbf{x},t)d^d\mathbf{x} - \int_\Omega \mathbf{u}^\nu(\mathbf{x},t')d^d\mathbf{x} \\ = & \;-\int_{t'}^t ds \int_{\partial\Omega}\left[\mathbf{u}^\nu(\mathbf{x},s)(\mathbf{u}^\nu(\mathbf{x},s)\cdot\hat{\mathbf{n}}) + p^\nu(\mathbf{x},s)\hat{\mathbf{n}} - \nu\frac{\partial\mathbf{u}^\nu}{\partial n}(\mathbf{x},s)\right]dA, \\ & \int_{\partial\Omega}\mathbf{u}^\nu(\mathbf{x},t)\cdot\hat{\mathbf{n}}\;dA = \int_{\partial\Omega}\mathbf{u}^\nu(\mathbf{x},t')\cdot\hat{\mathbf{n}}\;dA = 0 \end{aligned}$$

for all possible choices of Ω, t, t'. These equations just state that the momentum change in a bounded region Ω comes from flux of momentum across the surface $\partial\Omega$ and that there is no net flux of mass across the surface. Another equivalent formulation of the notion of notion of "weak" or distributional solution is that the "coarse-grained INS equations"

$$\partial_t \bar{\mathbf{u}}^\nu_\ell + \boldsymbol{\nabla}\cdot\overline{(\mathbf{u}^\nu\mathbf{u}^\nu)}_\ell = -\boldsymbol{\nabla}\bar{p}^\nu_\ell + \nu\triangle\bar{\mathbf{u}}^\nu_\ell, \quad \boldsymbol{\nabla}\cdot\bar{\mathbf{u}}^\nu_\ell = 0$$

should hold for all $\ell > 0$.

Importantly, Leray showed that his distributional solutions satisfy the following fundamental energy inequality for all $t \geq t_0$

$$\tfrac{1}{2}||\mathbf{u}^\nu(t)||^2_{L^2(V)} + \nu\int_{t_0}^t ds||\boldsymbol{\nabla}\mathbf{u}^\nu(s)||^2_{L^2(V)} \leq \tfrac{1}{2}||\mathbf{u}^\nu_0||^2_{L^2(V)}$$

This inequality states that the total energy at time t plus the integrated energy dissipation up to time t must be less than or equal to the initial energy $\frac{1}{2}||\mathbf{u}^\nu_0||^2_{L^2(V)}$. If the solutions are everywhere smooth ("strong"), then the inequality "$\leq$" becomes equality "$=$". However, if the solutions are singular ("weak"), then there may be strict inequality, which corresponds to "extra dissipation" due to the singularities, in addition to the viscous dissipation. Note that it is a consequence of this fundamental energy inequality that

$$\begin{aligned} \sup_{t_0\leq t\leq T} ||\mathbf{u}^\nu(t)||_{L^2(V)} &\leq ||\mathbf{u}_0||_{L^2(V)} < +\infty \\ \int_{t_0}^T dt||\boldsymbol{\nabla}\mathbf{u}^\nu(t)||^2_{L^2(V)} &\leq \frac{||\mathbf{u}_0||^2_{L^2(V)}}{2\nu} < +\infty \end{aligned}$$

for a fixed initial condition $\mathbf{u}_0$ with finite energy. From these estimates, some other basic bounds follow, such as

$$\int_{t_0}^T dt||\mathbf{u}^\nu(t)||^3_{L^3(V)} \leq (const.)\tfrac{L^2}{\nu}||\mathbf{u}_0||^3_{L^2(V)}.$$

This inequality means that Leray solutions are sufficiently regular that one can consider a local energy balance in spacetime. To construct his solutions, Leray considered the limit as $\ell \to 0$ of the modified equation

$$\partial_t\mathbf{u} + (\bar{\mathbf{u}}_\ell\cdot\boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla}p + \nu\triangle\mathbf{u}$$

where only the velocity $\bar{\mathbf{u}}_\ell = G_\ell * \mathbf{u}$ appearing in the advection term has been smoothed. Leray showed that the above equation has regular solutions $(\mathbf{u}^\star_\ell, p^\star_\ell)$, which lie in a bounded (and thus weakly compact) subset of $L^2([0,T], H^1(V))$. Hence, weak limits $\mathbf{u}^\star_\ell \to \mathbf{u}$ exist in this space, along subsequences of ℓ. Such $\mathbf{u}$ can be seen to be distributional solutions of the incompressible Navier-Stokes equation

$$\partial_t\mathbf{u} + \boldsymbol{\nabla}\cdot(\mathbf{u}\mathbf{u}) = -\boldsymbol{\nabla}p + \nu\triangle\mathbf{u}$$

and, since (by weak lower semi-continuity)

$$\liminf_{\ell\to 0} \int d^d x \int dt |\boldsymbol{\nabla}\mathbf{u}^\star_\ell(\mathbf{x},t)|^2 \psi(\mathbf{x},t) \geq \int d^d x \int dt |\boldsymbol{\nabla}\mathbf{u}(\mathbf{x},t)|^2 \psi(\mathbf{x},t)$$

for non-negative test functions ψ, also obeys the energy balance

$$\partial_t(\tfrac{1}{2}|\mathbf{u}|^2) + \boldsymbol{\nabla}\cdot\left[(\tfrac{1}{2}|\mathbf{u}|^2 + p)\mathbf{u} - \nu\boldsymbol{\nabla}(\tfrac{1}{2}|\mathbf{u}|^2)\right] = -D(\mathbf{u}) - \nu|\boldsymbol{\nabla}\mathbf{u}|^2$$

with $D(\mathbf{u}) \geq 0$. By smoothing the solution $\mathbf{u}$ one can write an energy balance also for $\bar{\mathbf{u}}_\ell$:

$$\begin{aligned}
\partial_t(\frac{1}{2}\mathbf{u}\cdot\bar{\mathbf{u}}_\ell) \quad &+ \quad \boldsymbol{\nabla}\cdot[(\frac{1}{2}\mathbf{u}\cdot\bar{\mathbf{u}}_\ell)\mathbf{u} + \frac{1}{2}(p\bar{\mathbf{u}}_\ell + \bar{p}_\ell\mathbf{u}) \\
&+ \quad \frac{1}{4}\overline{((|\mathbf{u}|^2\mathbf{u})}_\ell - \frac{1}{4}\overline{(|\mathbf{u}|^2)}\mathbf{u} - \nu\boldsymbol{\nabla}\frac{1}{2}\mathbf{u}\cdot\bar{\mathbf{u}}_\ell] \\
&= \quad -\frac{1}{4\ell}\int d^d\mathbf{r}\ \boldsymbol{\nabla}G_\ell(\mathbf{r})\cdot\delta\mathbf{u}(\mathbf{r})|\delta\mathbf{u}(\mathbf{r})|^2 - \nu\boldsymbol{\nabla}\mathbf{u} : \boldsymbol{\nabla}\bar{\mathbf{u}}_\ell
\end{aligned}$$

Taking the limit $\ell \to 0$, we recover the previous energy balance with an explicit expression for $D(\mathbf{u})$:

$$D(\mathbf{u}) = \lim_{\ell\to 0} \tfrac{1}{4\ell} \int d^d\mathbf{r}(\boldsymbol{\nabla}G)_\ell(\mathbf{r})\cdot\delta\mathbf{u}(\mathbf{r})|\delta\mathbf{u}(\mathbf{r})|^2$$

This formula makes it clear that $D(\mathbf{u})$ in the Navier-Stokes solution is connected with velocity increments. Note that, in general, in the presence of such singularities, the total Navier-Stokes dissipation is

$$D(\mathbf{u}^\nu) + \nu|\boldsymbol{\nabla}\mathbf{u}^\nu|^2$$

(where we have now added the supperscript ν to indicate the viscosity value) and in the $\nu \to 0$ limit it is this total dissipation which converges to the anomalous dissipation in the Euler solution

$$D(\mathbf{u}^\nu) + \nu|\boldsymbol{\nabla}\mathbf{u}^\nu|^2 \to D(\mathbf{u}).$$

For more details, see Duchon & Robert (2000)

General References on Leray Solutions of INS

An English translation of Leray's paper of 1934 is available at the Cornell website of mathematician Bob Terrell:

http://www.math.cornell.edu/~bterrell/leray.pdf

A key paper on "partial regularity" of Leray solutions, improving the earlier work of V. Scheffer (1977) is

> L. Cafarelli, R. Kohn & L. Nirenberg, "Partial regularity of suitable weak solutions of the Navier-Stokes equations," Commun. Pure Appl. Math. **35**, 771-831 (1982)

Many good textbooks presentation of the Leray theory are available. I'd recommend

> R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis* (AMS, 2001)

and

> R. Temam, *Navier-Stokes Equation & Nonlinear Functional Analysis* (SIAM, 1995)

which make nice connections with numerical analysis, and

> G. Gallavotti, *Foundations of fluid dynamics* (Springer, 2013)

which is a good discussion for (mathematically inclined) physicists.