1. (a) Show that the gradient of the Lagrangian map $X(\alpha, t)$ satisfies

$$\frac{\partial}{\partial t} \nabla_\alpha X = \nabla_\alpha X \cdot \nabla_x u.$$ 

Use this result to show that Cauchy’s vorticity formula $\omega(X(\alpha, t), t) = \Omega(\alpha) \cdot \nabla_\alpha X(\alpha, t)$ provides an explicit integration of the 3D Euler equation $D_t \omega = \omega \cdot \nabla_x u$.

(b) The **Weber velocity variable**

$$w(\alpha, t) \equiv \nabla_\alpha X(\alpha, t) \cdot v(\alpha, t)$$

is closely related to the Cauchy invariant. Establish the so-called Weber formulation of the 3D Euler equation:

$$\frac{\partial}{\partial t} w = \nabla_\alpha \left[ \frac{1}{2} |v|^2 - p_L \right],$$

where note that $p_L(\alpha, t) = p(X(\alpha, t), t)$ is the Lagrangian pressure.

(c) If $C$ is any fixed loop in the label space, show that

$$\oint_C d\alpha \cdot w(\alpha, t) = \oint_{C(t)} d\mathbf{x} \cdot u(\mathbf{x}, t)$$

where $C(t)$ is the image of $C$ under the Lagrangian flow $X(\alpha, t)$. Then use the result in part (b) to give another proof of the Kelvin circulation theorem.

(d) Show that Cauchy’s vorticity invariant is the curl of Weber’s velocity variable:

$$\Omega(\alpha) = \nabla_\alpha \times w(\alpha, t).$$

*Hint:* Define $\Omega^*(\alpha) \equiv \nabla_\alpha \times w(\alpha, t)$ and then calculate $\Omega^*(\alpha) \cdot \nabla_\alpha X(\alpha, t)$. You will find useful the result

$$\epsilon_{ijk} \frac{\partial X_m}{\partial \alpha_i} \frac{\partial X_n}{\partial \alpha_j} \frac{\partial X_k}{\partial \alpha_l} = \epsilon_{lnm},$$

which you should show follows from the Jacobian, $\partial(X_1, X_2, X_3)/\partial(\alpha_1, \alpha_2, \alpha_3) = 1$. 

2. (a) Show that
\[
\int_0^t ds \int_0^t ds' \ F\left( s' - s, \frac{s' + s}{2} \right) = \left[ \int_0^t d\tau \int_{\tau/2}^{t-\tau/2} dT + \int_0^0 d\tau \int_{-\tau/2}^{t+\tau/2} dT \right] f(\tau, T)
\]
with \( \tau = s' - s, \ T = (s' + s)/2. \)

(b) Use part (a) to show that, if the statistics of the Lagrangian velocity \( \mathbf{v}(\alpha, t) \) are stationary in time, then
\[
\langle |\delta \mathbf{X}(t)|^2 \rangle = \int_0^t d\tau \left( 1 - \frac{\tau}{t} \right) \langle \mathbf{v}(\tau) \cdot \mathbf{v}(0) \rangle.
\]

(c) Finally, show that
\[
\lim_{t \to \infty} \frac{\langle |\delta \mathbf{X}(t)|^2 \rangle}{2t} = D
\]
with \( D = \int_0^\infty d\tau \langle \mathbf{v}(\tau) \cdot \mathbf{v}(0) \rangle \), if the velocity auto-correlation function \( \langle \mathbf{v}(\tau) \cdot \mathbf{v}(0) \rangle \) is (absolutely) integrable in time.

3. (a) Suppose that \( \mathbf{u}(\mathbf{x}, t) \) for all times \( t \) is a smooth velocity field in a flow domain \( \Omega \) that is incompressible and has no flow across the boundary \( \partial \Omega \) of the domain: \( \nabla \cdot \mathbf{u} = 0 \) and \( \mathbf{n} \cdot \mathbf{u} |_{\partial \Omega} = 0. \) Explain why, for any function \( F(\mathbf{x}, t) \),
\[
\int_\Omega d^d \mathbf{x} F(\mathbf{x}, t) = \int_\Omega d^d \alpha F(\mathbf{X}(\alpha, t), t),
\]
where \( \mathbf{X}(\alpha, t) \) is the Lagrangian flow map generated by \( \mathbf{u}. \)

(b) Now suppose that \( \mathbf{u}(\mathbf{x}, t) \) is an incompressible velocity field on all of Euclidean space \( \mathbb{R}^d \) which, for simplicity, is absolutely bounded: \( |\mathbf{u}(\mathbf{x}, t)| \leq u_{\text{max}} < \infty. \) If \( \Omega_n \) is a sequence of domains \( \Omega_n \uparrow \mathbb{R}^d, \) then explain why, for any fixed time \( t, \)
\[
\lim_{n \to \infty} \frac{1}{|\Omega_n|} \int_{\Omega_n} d^d \mathbf{x} F(\mathbf{x}, t) = \lim_{n \to \infty} \frac{1}{|\Omega_n|} \int_{\Omega_n} d^d \alpha F(\mathbf{X}(\alpha, t), t),
\]
where \( |\Omega_n| \) is the \( d \)-dimensional volume of \( \Omega_n \).

(c) If statistics are defined by space-averaging, then show that the single-time, 1-point PDF’s of Eulerian velocity \( P(\mathbf{u}, t) \) and of Lagrangian velocity \( P(\mathbf{v}, t) \) are identical under either of the assumptions in (a) and (b). It may be helpful to consider the Fourier transform, or so-called characteristic function,
\[
\langle \exp(ik \cdot \mathbf{u}(\cdot, t)) \rangle = \int d^d \mathbf{u} e^{ik \cdot \mathbf{u}} P(\mathbf{u}, t)
\]
with average on the left over \( \mathbf{x}, \) and similarly for \( P(\mathbf{v}, t). \)
4. In this problem we consider the solution of Richardson’s equation for 2-particle relative diffusion in 3D spherical coordinates,

$$\partial_t P(\rho,t) = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left( \rho^2 K(\rho) \frac{\partial}{\partial \rho} P(\rho,t) \right),$$

with eddy-diffusivity $K(\rho) = k_0 \langle \varepsilon \rangle \frac{1}{3} \rho^{4/3}$.

(a) Define a new time-like variable $\tau = k_0 \langle \varepsilon \rangle \frac{1}{3} t$ and a time-dependent length-scale $L(\tau) = \tau^{3/2}$. Looking for solutions of the self-similar form

$$P(\rho,\tau) = \frac{1}{L^3(\tau)} Q \left( \frac{\rho}{L(\tau)} \right),$$

show that $Q(x)$ with $x = \rho/L(\tau)$ must satisfy the equation

$$\frac{9}{2} Q(x) + \frac{3}{2} x Q'(x) + \frac{10}{3} x^{1/3} Q'(x) + x^{4/3} Q''(x) = 0.$$

(b) Making the substitution $y = \frac{3}{2} x^{2/3}$, show that the resulting equation has an exact solution $Q(y) = \exp\left( -(3/2)y \right)$. From this infer that

$$P(\rho,t) = \frac{A}{(k_0 \langle \varepsilon \rangle \frac{1}{3} t)^{3/2}} \exp \left[ -\frac{9}{4} \frac{\rho^{2/3}}{k_0 \langle \varepsilon \rangle \frac{1}{3} t} \right]$$

is an exact solution of Richardson’s equation for any constant $A$.

(c) Show that the value of the constant to satisfy the normalization condition

$$4\pi \int_0^\infty \rho^2 d\rho P(\rho,t) = 1$$

is given by $1/A = 4\pi (2/3)^8 \Gamma(9/2)$ with $\Gamma(z) = \int_0^\infty dt t^{z-1} e^{-t}$. Explain why it is true, with this value of $A$, that $\lim_{t\to0^+} P(\rho,t) = \delta^3(\rho)$.

(d) Show for the solution in (b),(c) that

$$\langle \rho^2(t) \rangle \equiv 4\pi \int_0^\infty \rho^4 d\rho P(\rho,t) = g_0 \langle \varepsilon \rangle t^3,$$

with the Richardson constant $g_0 = (1144/81)k_0^3$. Hint: Recall that $\Gamma(z+1) = z\Gamma(z)$. 

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5. In this problem you will derive the *Ott-Mann-Gawędzki relation* which is sometimes described as the “Lagrangian analog of the 4/5th law.” This relation states that for homogeneous Navier-Stokes turbulence

\[ \frac{1}{2} \frac{d}{dt} \left| \langle |\delta \mathbf{v}(r, t)|^2 \rangle \right|_{t=0} \simeq -2 \langle \varepsilon \rangle, \quad \eta_K \ll r \ll L \]

where \( \mathbf{v}(\alpha, t) \) is the Lagrangian velocity with particles labelled at time \( t = 0 \), \( \delta \mathbf{v}(\mathbf{r}; \alpha, t) = \mathbf{v}(\alpha + \mathbf{r}, t) - \mathbf{v}(\alpha, t) \) is the Lagrangian velocity increment in label space, and \( \varepsilon = \nu |\nabla \mathbf{u}|^2 \) is the viscous energy dissipation.

(a) Show that

\[ \frac{1}{2} \frac{d}{dt} \langle |\delta \mathbf{v}(r, t)|^2 \rangle \bigg|_{t=0} = \langle \delta \mathbf{u}(\mathbf{r}) \cdot \delta \mathbf{a}(\mathbf{r}) \rangle \]

where \( \mathbf{u} \) is the Eulerian velocity field and \( \mathbf{a} = -\nabla p + \nu \triangle \mathbf{u} + \mathbf{f} \) the Eulerian acceleration field due to pressure gradient, viscosity and body force. (This means that the relation is only “instantaneously Lagrangian.”)

(b) Show that

\[ \langle \delta \mathbf{u}(\mathbf{r}) \cdot \delta \mathbf{a}(\mathbf{r}) \rangle = -2 \langle \varepsilon \rangle + \nu \nabla_r \langle |\delta \mathbf{u}(\mathbf{r})|^2 \rangle + \langle \delta \mathbf{u}(\mathbf{r}) \cdot \delta \mathbf{f}(\mathbf{r}) \rangle. \]

(c) Assuming the inertial-range scaling \( \langle |\delta \mathbf{u}(\mathbf{r})|^2 \rangle \sim u_{rms}^2 (r/L)^{\zeta_2} \) with \( \zeta_2 = 2/3 \), show

\[ \nu \nabla_r \langle |\delta \mathbf{u}(\mathbf{r})|^2 \rangle \sim \langle \varepsilon \rangle Re^{\frac{3}{2} \zeta_2 - \frac{5}{2}} \left( \frac{r}{\eta_K} \right)^{\zeta_2 - 2} \ll \langle \varepsilon \rangle, \quad r \gg \eta_K. \]

Recall that \( \langle \varepsilon \rangle \sim u_{rms}^3 / L. \)

(d) Assuming that the body force is smooth so that

\[ \langle |\delta \mathbf{f}(\mathbf{r})|^2 \rangle = O \left( f_{rms}^2 (r/L)^2 \right), \]

show that

\[ |\langle \delta \mathbf{u}(\mathbf{r}) \cdot \delta \mathbf{f}(\mathbf{r}) \rangle| = O \left( \langle \varepsilon \rangle \left( \frac{r}{L} \right)^{1+\zeta_2/2} \right) \ll \langle \varepsilon \rangle, \quad r \ll L. \]

Observe that \( \langle \varepsilon \rangle \leq u_{rms} f_{rms} \), but assume further that \( \langle \varepsilon \rangle \sim u_{rms} f_{rms} \).