1. (a) Use the “shift trick” to give a careful derivation of the formula

\[
(\nabla \times \mathbf{f}_\ell^S)_i = \epsilon_{ijk} \frac{1}{\ell^2} \times \left[ \int d^3r \left( \partial_m \partial_k G_\ell(r) \delta u_j(r) \delta u_m(r) \right) - \int d^3r \left( \partial_m \partial_k G_\ell(r) \delta u_j(r) \delta u_m(r') \right) \int d^3r' G_\ell(r') \delta u_m(r) \right] - \int d^3r \left( \partial_m G_\ell(r) \delta u_m(r) \right) \int d^3r' (\partial_k G_\ell(r') \delta u_j(r')) \].
\]

(b) Use the result in (a) to derive the rigorous estimate \(\nabla \times \mathbf{f}_\ell^S = O(\delta u^2(\ell)/\ell^2)\).

2. (a) Consider the balance equation for the large-scale enstrophy:

\[
D_{\ell t} \left( \frac{1}{2} |\omega_\ell|^2 \right) + \nabla \cdot (\omega_\ell \times \mathbf{f}_\ell^S) - \nu \triangle \left( \frac{1}{2} |\omega_\ell|^2 \right) = \omega_\ell^T \mathbf{S}_\ell \omega_\ell + (\nabla \times \omega_\ell) \cdot \mathbf{f}_\ell^S - \nu |\nabla \omega_\ell|^2 + \omega_\ell \cdot (\nabla \times \mathbf{f}_\ell^B).
\]

Explain briefly how to derive the following bounds on each of the terms:

\[
D_{\ell t} \left( \frac{1}{2} |\omega_\ell|^2 \right), \nabla \cdot (\omega_\ell \times \mathbf{f}_\ell^S), \omega_\ell^T \mathbf{S}_\ell \omega_\ell, (\nabla \times \omega_\ell) \cdot \mathbf{f}_\ell^S = O \left( \frac{\delta u^3(\ell)}{\ell^3} \right),
\]

\[
\nu \triangle \left( \frac{1}{2} |\omega_\ell|^2 \right), \nu |\nabla \omega_\ell|^2 = O \left( \frac{\delta u^3(\ell)}{Re \ell \cdot \ell^3} \right), \omega_\ell \cdot (\nabla \times \mathbf{f}_\ell^B) = O \left( \frac{\delta u(\ell)}{\ell} \|\nabla \mathbf{f}_\ell^B\|_{\infty} \right).
\]

(b) The balance equation of Tennekes & Lumley, eq.(3.3.36), is somewhat different than that in part (a). In particular, it contains a cross-term contribution to vortex-stretching which should satisfy the following heuristic bound:

\[
\omega_\ell^T \tau_\ell (S_{ij}, \omega_j) = O^* \left( \frac{\delta u^3(\ell)}{\ell^3} \right).
\]

Give the most careful justification that you can for this estimate.
3. This problem considers the scaling properties of the energy dissipation $\varepsilon = 2\nu S^2$ and the “pseudo-dissipation” $\varepsilon^\omega = \nu \omega^2$.

(a) Derive the formula
\[ \varepsilon^\omega_\ell = \tilde{\varepsilon}_\ell - \frac{2\nu}{\ell^2} \int d^dr \left( \partial_i \partial_j G \right)_\ell (r) \delta u_i (r) \delta u_j (r) \]
and use it to show that $|\varepsilon^\omega_\ell - \varepsilon_\ell| \leq C\nu \delta u^2 (\ell) / \ell^2$ for an absolute constant $C$.

(b) Use the Hölder inequality to establish $(x+y)^p \leq 2^{p-1} (x^p + y^p)$ for $x, y \geq 0$ and any $p \geq 1$.

(c) Use parts (a), (b) to show that for any $p \geq 3$
\[ 2^{1-p/3} \varepsilon^\omega_\ell + O \left( \nu \frac{\delta u^2 (\ell)}{\ell^2} \right)^{p/3} \leq \varepsilon_\ell^{p/3} \leq 2^{p/3-1} \varepsilon^\omega_\ell + O \left( \nu \frac{\delta u^2 (\ell)}{\ell^2} \right)^{p/3} \]

(d) The refined similarity hypothesis (RSH) may be stated as the property that $\varepsilon_\ell = W \cdot \delta u^3 (\ell) / \ell$, where $W$ is a (universal) random variable independent of $\delta u (\ell)$. Assuming the RSH for the true energy dissipation, show that
\[ C^- \frac{S_p (\ell)}{\ell^{p/3}} + O \left( \nu \frac{S_{2p/3} (\ell)}{\ell^{2p/3}} \right) \leq \langle \varepsilon^\omega_\ell \rangle^{p/3} \leq C^+ \frac{S_p (\ell)}{\ell^{p/3}} + O \left( \nu \frac{S_{2p/3} (\ell)}{\ell^{2p/3}} \right) \]
where $S_p (\ell) = \langle \delta u^p (\ell) \rangle$ and $C^- < C^+$ are constants.

(e) Use the estimate in (d) to show that a scaling law $\langle \varepsilon^\omega_\ell \rangle^{p/3} \sim \langle \varepsilon \rangle^{p/3} (\ell / L)^{\zeta_p}$ can hold at high Reynolds number for $p \geq 3$ only with $\tau_{p/3} = \zeta_p - p/3$, where $\zeta_p$ is the scaling exponent of $S_p (\ell)$. In particular, show that this scaling law will hold down to a length-scale $\ell < L$ if
\[ \left( \frac{u_0 \ell}{\nu} \right)^{p/3} \left( \frac{\ell}{L} \right)^{\zeta_p - \zeta_{2p/3}} \gg 1. \]

4. This problem studies the Nonlinear Model approximation to the vortex force
\[ f_i^{V^*} = \frac{1}{3} C_2 \ell^2 \varepsilon_{ijkl} u_{j,l} \tilde{\omega}_{k,l}. \]
For simplicity, we drop the subscript length-scale $\ell$.

(a) Show that
\[ f_i^{V^*} = \frac{1}{3} C_2 \ell^2 \varepsilon_{ijkl} \tilde{S}_{j,l} \tilde{\omega}_{k,l} + \frac{1}{12} C_2 \ell^2 \partial_i (|\tilde{\omega}|^2). \]
(b) Obtain from (a) the approximation for the torque on a loop $C$:

$$\Gamma^*(C) = -\frac{1}{3} C_2 \epsilon_{ijk} \ell^2 \oint_C \bar{S}_{ji} \bar{\omega}_{k,l} dx_l.$$

(c) Derive the alternative expression

$$f^{V^*} = \frac{1}{3} C_2 \ell^2 \nabla \cdot (\bar{S} \times \bar{\omega}) + \frac{1}{6} C_2 \ell^2 (\bar{\omega} \cdot \nabla) \bar{\omega}.$$

*Hint:* Calculate $(\bar{S} \times \bar{\omega})_{ji,j}$.

(d) Use the result in (c) to rederive the Nonlinear Model approximation for the mean helicity flux:

$$\langle \Lambda^*_x \rangle = -2 \langle \bar{\omega} \cdot f^{V^*} \rangle = \frac{2}{3} C_2 \ell^2 \langle \nabla \omega : (\bar{S} \times \bar{\omega}) \rangle.$$

(e) Show that the vorticity contribution to the vortex force $f^{vort} = \frac{1}{6} C_2 \ell^2 (\bar{\omega} \cdot \nabla) \bar{\omega}$ from (c) can be written as $f^{vort} = \alpha^{vort} \bar{\omega} + \Delta u^{vort} \times \bar{\omega}$ with

$$\alpha^{vort} = \frac{1}{6} C_2 \ell^2 \frac{\partial |\bar{\omega}|}{\partial s}, \quad \Delta u^{vort} = \frac{1}{6} C_2 \ell^2 \kappa |\bar{\omega}| \hat{b},$$

where $s$ is arclength along the vortex-line, $\kappa$ is the local curvature of the line, and $\hat{t}, \hat{n}, \hat{b}$ is the Frenet-Serret frame of unit tangent, normal and binormal vectors.

*Hint:* Use $\bar{\omega} = |\bar{\omega}| \hat{t}$, $\bar{\omega} \cdot \nabla = |\bar{\omega}| \partial/\partial s$, and the Frenet-Serret equations.