Problem 1. (a) We must show that the Besov space embedding theorem (BSET) is equivalent to \( \star \) or \( \overset{\Rightarrow}{\leftrightarrow} \star \).

We first prove that \( \overset{\Rightarrow}{\leftrightarrow} \star \). Consider any index \( s \) such that

\[ s < \sigma_{p'} + d\left(\frac{1}{p} - \frac{1}{p'}\right), \quad p \geq p' \]

There must exist some \( \varepsilon > 0 \) such that

\[ s < (\sigma_{p'} - \varepsilon) + d\left(\frac{1}{p} - \frac{1}{p'}\right). \]

Since \( f \in B^{\sigma_{p'} - \varepsilon, \infty}_{p'} \) by the definition of the maximal Besov exponent \( \sigma_{p'} \), it then follows by the BSET that

\[ f \in B^{\sigma_{p'} - \varepsilon, \infty}_{p'} \subseteq B^{s, \infty}_p, \quad \forall s < \sigma_{p'} + d\left(\frac{1}{p} - \frac{1}{p'}\right) \]

Hence, by definition of the maximal Besov exponent \( \sigma_p \),

\[ \sigma_p \geq \sigma_{p'} + d\left(\frac{1}{p} - \frac{1}{p'}\right) \]

or

\[ \sigma_p - \frac{d}{p} \geq \sigma_{p'} - \frac{d}{p'} \]

which is \( \star \)!
We next prove that $X \Rightarrow \text{BSET}$. Suppose therefore that for $p \geq p'$

$$s < s' + d\left( \frac{1}{p} - \frac{1}{p'} \right) \quad (*)$$

and take $f \in B^{s',\infty}_{p'}$. By the definition of the maximal Besov exponent $\sigma_p$, it follows that $s' \leq \sigma_p$. Hence $(*)$ implies that

$$\sigma_p - \frac{d}{p} \geq \sigma_{p'} - \frac{d}{p'} \geq s' - \frac{d}{p'},$$

or

$$\sigma_p \geq s' + d\left( \frac{1}{p} - \frac{1}{p'} \right).$$

Thus, $(*)$ implies that $s < \sigma_p$ and then, by the definition of the maximal Besov index $\sigma_p$, $f \in B^{s,\infty}_p$.

We see that when $(*)$ holds then

$$f \in B^{s',\infty}_{p'} \Rightarrow f \in B^{s,\infty}_p,$$

or, equivalently,

$$B^{s',\infty}_{p'} \subset B^{s,\infty}_p \quad \text{for} \quad p \geq p', \quad s - \frac{d}{p} < s' - \frac{d}{p'}$$

This is exactly BSET!  

QED
(b) Because the energy spectrum is the Fourier transform of the velocity correlation function (by the Wiener-Khintchine theorem), it follows that \( n = 1 + \frac{5}{2} \).

Thus,

\[
\sigma_2 = \frac{5}{3} \implies \sigma_2 > \frac{5}{6}
\]

for \( d = 3 \). Hence, \( \sigma_2 > \frac{5}{6} \) implies that

\[
\sigma_3 > \sigma_2 - \frac{1}{2} > \frac{5}{6} - \frac{1}{2} = \frac{1}{3} \implies \sigma_3 > 1 \quad \text{QED}
\]

Problem 2. (a) In the multifractal model

\[
\eta_h \sim (\text{const.}) \frac{L(Re)}{1 + h}
\]

whereas the K41 value corresponds to \( h = \frac{1}{3} \), \( \eta \sim (\text{const.}) L(Re)^{-\frac{3}{4}} \).

Thus,

\[
\eta_h/\eta \sim (\text{const.}) (Re)^{\frac{3}{4} - \frac{1}{1 + h}} = (\text{const.}) (Re)^{\frac{3h-1}{4(1+4)}}
\]

(b) We recall the result of Corrsin (1959) that

\[
\eta / \text{lmsf} \sim (\text{const.}) (Re)^{\frac{1}{4}} / \text{Ma}
\]

Putting this together with (a) gives (cont'd)
with
\[
\frac{3h-1}{4(1+h)} + \frac{1}{q} = \frac{(3h-1) + (1+h)}{4(1+h)} = \frac{4h}{4(1+h)} = \frac{h}{1+h}
\]

that
\[
\eta \frac{\lambda}{\text{Re}} = \eta \frac{\lambda_{\text{wall}}}{\text{Re}} = (\eta \frac{\lambda_{\text{wall}}}{\text{Re}}) \frac{(3h-1)}{4(1+h)} \frac{\text{Re}}{\text{Ma}} \sim (\text{const.}) \frac{(\text{Re})^{1/4}}{\text{Ma}}
\]

Problem 3, (a) According to our criterion, \( \varepsilon_r(x) \) is an \underline{inertial} range quantity. For example, if \( \varepsilon_r(x) = 2(\pi)^2 |S(x)|^2 \) converges as \( \nu \to 0 \) to a measure \( \varepsilon \), as discussed in our proof of the \( \psi \) law, then
\[
\varepsilon_r(x) = \frac{3}{4\pi r^3} \int_{|r|<r} d^3 r \varepsilon(x+r)
\]

where \( B_r(x) \) is the ball of radius \( r \) at the point \( x \) and this is a \underline{bounded} function of the space point \( x \) at almost every time \( t \). For instance, in free decay
\[
\sum_{t} \varepsilon(B_r(x), t) \leq \frac{1}{2} \| W_0 \|_2^2
\]

Since one cannot dissipate more energy than present initially.
(b) From the condition of constant flux

\[ \langle ||T_r||_1 \rangle = \langle \varepsilon \rangle \] independent of \( r \)

we obtain

\[ \langle \varepsilon \rangle = \frac{1}{\langle ||T_r||_1 \rangle} \]

\[ = \frac{\|T_r\|_1}{\|T_r\|_p}, \quad p \geq 1 \]

\[ \sim \text{const.} \langle \varepsilon \rangle \left( \frac{r}{L} \right)^{\frac{\tau_p}{p}} \]

This inequality is only consistent for \( r \ll L \) if \( \tau_p \leq 0 \).

Furthermore, from \( T_r = -S_r : T_r \) we get that for \( p \geq 1 \)

\[ \|T_r\|_p \leq \|S_r\|_2 \|T_r\|_2^{1/2} \]

\[ = O\left( r^{3\delta_{3p}} \right) \cdot O\left( r^{2\delta_{3p}} \right) \]

\[ = O\left( r^{3\delta_{3p}-1} \right) \]

Using bounds derived in class, taking \( p \)-th powers of both sides, this is equivalent to

\[ \langle ||T_r||_p^p \rangle = O\left( r^{5\delta_{3p}-p} \right) \]

If the left-hand-side scales as \( \langle \varepsilon \rangle^p \left( \frac{r}{L} \right)^{\tau_p} \), then, again for \( r \ll L \) this can only be consistent if

\[ \tau_p \geq 5\delta_{3p} - p \quad \text{for} \quad p \geq 1 \]

\[ \alpha \]

\[ 5\delta_p \leq \frac{p}{3} + \frac{\tau_p}{3} \quad \text{for} \quad p \geq 3 \]
(c) Using the notations

\[ u_L(x) = \tilde{r} \cdot u(x), \quad u_L'(x) = \tilde{r} \cdot u(x+\tau) \]

we see that

\[ \delta u_L(x) = u_L' - u_L, \quad u_L^{av}(x) = \frac{u_L' + u_L}{2} \]

or, inversely,

\[ u_L = u_L^{av} + \frac{1}{2} \delta u_L, \quad u_L = u_L^{av} - \frac{1}{2} \delta u_L. \]

Thus,

\[
\langle (u_L')^3 \rangle = \langle (u_L^{av})^3 \rangle + \frac{3}{2} \langle (u_L^{av})^2 \delta u_L \rangle + \frac{3}{4} \langle u_L^{av} \delta u_L \rangle^2 \\
+ \frac{1}{8} \langle \delta u_L \rangle^3
\]

\[
\langle (u_L)^3 \rangle = \langle (u_L^{av})^3 \rangle - \frac{3}{2} \langle (u_L^{av})^2 \delta u_L \rangle + \frac{3}{4} \langle u_L^{av} \delta u_L \rangle^2 \\
- \frac{1}{8} \langle \delta u_L \rangle^3
\]

Subtracting these two equal quantities gives

\[
3 \langle \delta u_L (u_L^{av})^2 \rangle = -\frac{1}{4} \langle (\delta u_L)^3 \rangle
\]

\[
\langle \delta u_L (u_L^{av})^2 \rangle = -\frac{1}{12} \langle (\delta u_L)^3 \rangle.
\]

Using the 4th moment, this becomes

\[
\langle \delta u_L (u_L^{av})^2 \rangle = \frac{\langle \delta \rangle}{15}.
\]
(d) Substituting the relation

\[ \delta u_L = W_r (\varepsilon_r)^{1/3} \]

gives

\[ \langle [u_L^\omega(r)]^2 W_r \varepsilon_r^{1/3} \rangle \sim \langle \varepsilon \rangle^{2/3}/15. \]

The average on the left-hand side can be written as

\[ \langle [u_L^\omega(r)]^2 W_r \varepsilon_r^{1/3} \rangle = \int du_L^\omega \int d\varepsilon_r \int dW_r \ P(W_r | u_L^\omega, \varepsilon_r) \ P(u_L^\omega, \varepsilon_r) \ W_r \ [u_L^\omega(j)]^2 \varepsilon_r^{1/3} \]

Suppose that

\[ P(W_r | u_L^\omega, \varepsilon_r) \equiv P(W_r) \]

independent of \( u_L^\omega \) and \( \varepsilon_r \) for \( Re_r \gg 1 \). Taking the limit \( \nu \to 0 \) one then obtains that

\[ \langle [u_L^\omega]^2 W_r \varepsilon_r^{1/3} \rangle = \int du_L^\omega \int d\varepsilon_r \int dW_r \ P(W_r) \ P(u_L^\omega, \varepsilon_r) \ W_r \ [u_L^\omega(j)]^2 \varepsilon_r^{1/3} \]

\[ = \langle W_r \rangle \langle (u_L^\omega(j))^2 \varepsilon_r^{1/3} \rangle \]

However, a similar argument applied to

\[ 0 = \langle \delta u_L \rangle = \langle W_r (\varepsilon_r)^{1/3} \rangle \]

(cont'd)
0 = \langle W_r^L \rangle \langle (\varepsilon_r r)^{1/3} \rangle \\
and thus \\
\langle W_r^L \rangle = 0.

However, this is clearly inconsistent with the result

\[
\frac{< \varepsilon_r >^{2/3}}{15} = \langle \left[ u_L^w(r) \right]^2 W_r^L \varepsilon_r^{1/3} \rangle \\
= \langle W_r^L \rangle \langle \left[ u_L^w(r) \right]^2 \varepsilon_r^{1/3} \rangle.
\]

Thus, we can conclude that \( P(W_r^L | U_L^w, \varepsilon_r) \) cannot be independent of both \( U_L^w(r) \) and \( \varepsilon_r \) for \( Re_P >> 1 \).

Hosokawa plausibly argues that the PDF becomes independent of \( \varepsilon_r \) for \( Re_P >> 1 \) but remains dependent on \( U_L^w(r) \).

Remark: Note that this contradiction cannot be obtained if one applies the RSH to \underline{absolute} velocity increments, i.e., considers

\[
|\delta u_L| = W_r^L (\varepsilon_r r)^{1/3}
\]

since it is not true that

\[
0 = \langle |\delta u_L| \rangle
\]

and this was the essential ingredient used above.
Problem 4. (a) Using the same formula as in the proof of pointwise locality

\[ S\bar{u}_\Delta(r; x) = \int_0^1 d\theta \ (r \cdot \nabla) \bar{u}_\Delta(x + \theta r) \]

we get that

\[ \| S\bar{u}_\Delta(r) \|_p \leq \int_0^1 d\theta \ |r| \ \| \nabla\bar{u}_\Delta(\cdot + \theta r) \|_p \]

\[ = r \cdot \| \nabla\bar{u}_\Delta \|_p. \]

However, it was proved in class some time ago that

\[ \| \nabla\bar{u}_\Delta \|_p = O(\Delta^{p-1}) \]

where \( \| S\bar{u}(r) \|_p \sim (\text{const}) r^{p\Delta} \). Just to recall the proof, we remind that it uses the identity

\[ \nabla\bar{u}_\Delta(x) = -\frac{1}{\Delta} \int d^d r \ (\nabla \Delta)(r) S\bar{u}(r; x) \]

so that

\[ \| \nabla\bar{u}_\Delta \|_p \leq \frac{1}{\Delta} \int d^d r \ |(\nabla \Delta)(r)| \ \| S\bar{u}(r) \|_p \]

\[ \leq \frac{\text{const}}{\Delta} \int d^d r \ |(\nabla \Delta)(r)| \ |r|^{p\Delta} \]

\[ = O(\Delta^{p-1}) \]

since we can assume that \( \int d^d r \ |(\nabla \Delta)(r)| \ |r|^{p\Delta} < +\infty \). Putting the two estimates together we obtain

\[ \| S\bar{u}_\Delta(r) \|_p = O(r^{p\Delta} \Delta^{p-1}). \]
Because
\[ \| \delta w(r) \|_p \sim (\text{const.}) r^{-\kappa} \]
we obtain further that
\[ \| \delta w_\Delta(r) \|_p \leq \| \delta w(r) \|_p \cdot O\left(\left(\frac{r}{\Delta}\right)^{-\kappa} \right), \]
which is $L^\kappa$-locality of the velocity increment in $L^\kappa$-norm sense.

The second estimate is even simpler, because
\[ u_\delta'(x) = -\int d^d r \, G_\delta(r) \delta w(r; x) \]
gives
\[ \| u_\delta' \|_p \leq \int d^d r \, G_\delta(r) \| \delta w(r) \|_p \]
\[ \leq \text{const.} \int d^d r \, G_\delta(r) \| r \|^{-\kappa} \]
\[ = O(\delta^{-\kappa}) \]
assuming only that \( \int d^d r \, G(r) \| r \|^{-\kappa} < \infty \). Since
\[ \delta w_\delta'(r; x) = u_\delta'(x+r) - u_\delta'(x), \]
we can just use the triangle inequality to get
\[ \| \delta w_\delta'(r) \|_p \leq \| u_\delta'(x+r) \|_p + \| u_\delta' \|_p \]
\[ = 2 \| u_\delta' \|_p = O(\delta^{-\kappa}). \]
Again using that
\[ \| \delta u(r) \|_p \sim (\text{const.}) r^{\sigma_p} \]
we get that
\[ \| \delta u^\prime(r) \|_p = \| \delta u(r) \|_p \cdot O \left( \frac{\delta}{r} \right)^{\sigma_p} \]
which is **UV-locality** of the velocity increment in \( L_p \)-norm sense.

(b) Yes, \( L_p \)-locality bounds hold for the energy flux of the Burgers equation
\[ \mathcal{L} (u,u,u) = -\frac{1}{2} (\partial_x u) \mathcal{L} (u,u) \]
in any \( L_p \)-norm with \( 1 \leq p < +\infty \). Since "Burgers turbulence" or "Burgulence" has
\[ \| \delta u(r) \|_p \sim (\text{const.}) r^{\sigma_p} \]
with
\[ \sigma_p = \begin{cases} \frac{1}{p} & p < 1, \\ \frac{1}{p} & p > 1, \end{cases} \]
we see that \( 0 < \sigma_p < 1 \) for all \( 1 < p < \infty \). Using the Hölder inequality we can therefore obtain estimates such as **IR-locality**
\[ \| \mathcal{L} (\bar{u}_\delta, u,u) \|_p = \| \mathcal{L} (u,u,u) \|_p \cdot O \left( \frac{\delta}{\bar{\lambda}} \right)^{-\sigma_p} \]
or **UV-locality**
\[ \| \mathcal{L} (u^\prime\delta, u,u) \|_p = \| \mathcal{L} (u,u,u) \|_p \cdot O \left( \frac{\delta}{\bar{\lambda}} \right)^{-\sigma_p} \]
with any \( 1 < p < +\infty \).
Here we have assumed that

$$\| T^I(u, u', u') \|_p \sim (\text{const.}) \ell^{\frac{3\delta_p}{3\delta_p - 1}}$$

i.e., a Kraichnan-type RSH for Burgulence. If that is not true, we still obtain estimates like

$$\| T^I(u, u, u) \|_p = O(\ell^{\frac{2\delta_p}{3\delta_p} + 1} \delta_p^{-1})$$

which goes to zero for \( \Delta \gg \ell \), and

$$\| T^I(u, u', u, u') \|_p = O(\ell^{\frac{\delta_p}{3\delta_p - 1}} \delta_p^{-1})$$

which goes to zero for \( \delta \ll \ell \).

Note that for \( p > 1 \), \( \delta_p = \frac{1}{3}p \ll 1 \). Thus, the IR bounds became better, but the UV bounds became worse. In particular, UV-locality might not be uniformly valid in space for Burgulence, i.e., in the vicinity of the shock itself! A more refined estimation is necessary to see whether locality might still hold (at least marginally) in the vicinity of the shock. On the other hand, we see that Kraichnan's assertion is essentially correct: coherent discontinuous structures like shocks have local transfer. E.g., in a space average sense (\( p = 1 \)), energy flux is local since \( \delta_3 = \frac{1}{3} \).

Note that it is not quite the spectral exponent that determines locality in the mean, since that corresponds to \( \delta_2 \), or \( n = 1 + 2\delta_2 = 1 + \frac{2}{3} \). But Kraichnan was on the right track!