Homework #1 - Solutions

Problem 1. (b) Assuming that

$$h^*_t = \lim_{\rho \to 0} \frac{\ln \delta u(\rho)}{\ln (\rho/L)}$$

exists, we show that

$$h = \liminf_{\rho \to 0} \frac{\ln |\delta u(\rho)|}{\ln (\rho/L)} \leq h^*_t.$$

In fact, for \( r < \rho < L \),

$$\ln (\rho/L) < \ln (\rho/L) < 0.$$

By continuity of \( \delta u(\rho) \to 0 \) as \( \rho \to 0 \), so that \( |\delta u(\rho)| < 1 \)
when \( r < \rho_0 \) for some \( \rho_0 \) and thus, \( \ln |\delta u(\rho)| < 0 \). In that case,

$$\frac{\ln |\delta u(\rho)|}{\ln (\rho/L)} < \frac{\ln |\delta u(\rho)|}{\ln (\rho/L)}$$

for all \( r < \rho < \rho_0 \). Taking the infimum of both sides

$$\inf_{r < \rho} \frac{\ln |\delta u(\rho)|}{\ln (\rho/L)} < \inf_{r < \rho} \frac{\ln |\delta u(\rho)|}{\ln (\rho/L)} = \frac{\ln \delta u(L)}{\ln (L/L)},$$

using the fact that \( \ln (r/L) < 0 \) and the definition

$$\delta u(L) \equiv \sup_{\rho < L} |\delta u(\rho)|.$$

Taking the limit as \( L \to 0 \) gives

$$h = \liminf_{\rho \to 0} \frac{\ln |\delta u(\rho)|}{\ln (\rho/L)} \leq \lim_{L \to 0} \frac{\ln \delta u(L)}{\ln (L/L)} = h^*_t.$$
(c) We now show the opposite inequality that

\[ h_x \leq h. \]

If \( h_x = \lim_{l \to 0} \frac{\ln \delta u(r)}{\ln(\ell/L)} \) exists, then \( \forall \varepsilon > 0, \exists \delta > 0 \) such that \( \ell < \delta \Rightarrow \)

\[ h_x - \varepsilon < \inf_{r < \ell} \frac{\ln |\delta u(r)|}{\ln(\ell/L)} < h_x + \varepsilon. \]

This is equivalent to the statements that

\( \forall r < \ell, h_x - \varepsilon < \frac{\ln |\delta u(r)|}{\ln(\ell/L)} \)

and

\( \exists r < \ell, \frac{\ln |\delta u(r)|}{\ln(\ell/L)} < h_x + \varepsilon, \)

or, using \( \ln(\ell/L) < 0 \),

\( \forall r < \ell, |\delta u(r)| \leq \left( \frac{\ell}{L} \right)^{h_x - \varepsilon} \)

and

\( \exists r < \ell, |\delta u(r)| \geq \left( \frac{\ell}{L} \right)^{h_x + \varepsilon}. \)

By continuity of \( u \), \( |\delta u(r)| \leq \left( \frac{\ell}{L} \right)^{h_x - \varepsilon} \) holds as well for all \( r < \ell \), just by taking the limit as \( r \to \ell \). We thus obtain, in particular, that \( \forall \ell < \delta \)

\[ |\delta u(\ell)| \leq \left( \frac{\ell}{L} \right)^{h_x - \varepsilon} \]

with \( \ell = |\mathbf{x}| \), so that \( u \in C^{h_x - \varepsilon} \). Since \( h \) is the \underline{maximal} Hölder exponent, it follows that

\[ h_x - \varepsilon \leq h. \]

Since \( \varepsilon > 0 \) is arbitrary, we get finally that \( h_x \leq h \), as required.
Problem 2. (a) The Cantor set $K$ can be written as a disjoint union

$$K = \tilde{K}_0 \cup \tilde{K}_2$$

where

$$\tilde{K}_0 = K \cap [0, \frac{1}{3}], \quad \tilde{K}_2 = K \cap [\frac{2}{3}, 1].$$

We then note that

$$3K = K_0 \cup K_2$$

where

$$K_0 = 3\tilde{K}_0 = (3K) \cap [0, 1]$$

$$K_2 = 3\tilde{K}_2 = (3K) \cap [2, 3].$$

By the iterative construction of $K$ it is clear that, in fact,

$$K_0 = K \quad \text{and} \quad K_2 = K + 2,$$

which are just translations of $K$. Graphically,

Thus,

$$r = 3, \quad c = 2$$

so that

$$D_5(K) = \frac{\log 2}{\log 3}.$$
Of course, this agrees with $D_B(k) = \log 2 / \log 3$, illustrating the general theorem of Falconer that $D_S(S) = D_B(S)$ for any self-similar set.

(6) The Koch curve $C$ may likewise be written as

$$C = \tilde{C}_0 \cup \tilde{C}_1 \cup \tilde{C}_2 \cup \tilde{C}_3$$

as illustrated below:

These are non-intersecting if we adopt the convention, e.g., that the leftmost endpoint of $C$ is open. Then,

$$3C = C_0 \cup C_1 \cup C_2 \cup C_3 \quad \text{w/} \quad C_i = 3\tilde{C}_i$$

where

$$C_0 = C$$

(cont'd)
\[ C_1 = R_{60^\circ}(C) + (1,0) \]
\[ C_2 = R_{-60^\circ}(C) + (1,1) \]
\[ C_3 = C + (2,0) \]

where \( R_{\theta} \) denotes rotation by angle \( \theta \), i.e., \( r=3, c=4, D_3(C) = \frac{\log 4}{\log 3} \).

(c) Each elementary step in the construction of the Cantor set \( K \)

\[ \begin{array}{c}
\hline
\hline
\hline
\end{array} \quad \Rightarrow \quad \begin{array}{c}
\hline
\hline
\hline
\end{array} \]

reduces the length by a factor of \( \frac{2}{3} \). Since \( K_0 = [0,1] \) has length = 1, one obtains that

\[ L(K_n) = \left( \frac{2}{3} \right)^n \]

and

\[ L(K) = \lim_{n \to \infty} L_n(K_n) = 0. \]

Likewise, each elementary step in the construction of the Koch curve

\[ \begin{array}{c}
\hline
\hline
\hline
\end{array} \quad \Rightarrow \quad \begin{array}{c}
\hline
\hline
\hline
\end{array} \]

increases the length by a factor of \( \frac{4}{3} \). Since, again, \( C_0 = [0,1] \)

(cont'd)
one obtains that
\[ L(C_n) = \left( \frac{4}{3} \right)^n \]
and
\[ L(C) = \lim_{n \to \infty} L(C_n) = +\infty. \]

By the definition of $D_H(S)$ as a critical dimension, we see that
\[ 0 = L(K) = H^1(K) \Rightarrow D_H(K) < 1 \]
\[ \infty = L(C) = H^1(C) \Rightarrow D_H(C) > 1. \]

Problem 3. (a) We shall calculate the co-dimension $K(h)$ using
\[ K(h) = \sup_{\theta} \frac{\theta}{5_in} - \frac{1}{2} h. \]

Since $5_in = \min \{3_p, 2^\theta \}$, this becomes
\[ K(h) = \max_{\theta} \left( \sup_{p < 1} p(1-h), \sup_{p \geq 1} (1-ph)^\theta \right). \]

We first consider $h \in [0,1]$, so that $1-h \in [0,1]$. In that case,
\[ \sup_{p < 1} p(1-h) = 1-h \text{ for } p=1 \]
\[ \sup_{p \geq 1} (1-ph) = 1-h \text{ also for } p=1 \]
so that
\[ K(h) = \max \{1-h, (1-h)^2 \} = 1-h. \]
If, on the other hand, $b < 0$ then $1 - b > 1$, so that

$$\sup_{p<1} p(1-h) = 1-h \quad \text{for } p = 1$$

$$\sup_{p>1} (1-pb) = +\infty \quad \text{for } p = \infty$$

and thus,

$$R(h) = \max \{ 1-h, \infty \} = \infty.$$ 

Finally, suppose that $b > 1$ so that $1 - b < 0$. Therefore,

$$\sup_{p<1} p(1-h) = +\infty \quad \text{for } p = -\infty$$

$$\sup_{p>1} (1-pb) = 1-h \quad \text{for } p = 1$$

and thus,

$$R(h) = \max \{ \infty, 1-h \} = \infty.$$ 

Putting together all these results, we obtain from $\overline{D}(h) = 1-R(h)$ that

$$\overline{D}(h) = \begin{cases} 1 - (1-h) & h \in [0,1] \\ 1 - \infty & \text{otherwise} \end{cases}$$

$$= \begin{cases} h & h \in [0,1] \\ -\infty & \text{otherwise} \end{cases}$$
The problem is that

$$D_{\text{bifractal}}(h) = \begin{cases} 0 & h = 0 \\ 1 & h = 1 \\ -\infty & \text{o.w.} \end{cases}$$

is not concave! Concavity requires that for any $h \in [0,1]$

$$D(h) = D((1-h) \cdot 0 + h \cdot 1)$$

$$\geq (1-h)D(0) + hD(1)$$

$$= (1-h) \cdot 0 + h \cdot 1$$

$$= h$$

if $D(0)=0$, $D(1)=1$. However, for all $0 < h < 1$,

$$-\infty = D_{\text{bifractal}}(h) < h,$$

violating concavity. On the other hand, the Legendre transform

$$\overline{D}(h) = \inf_p \left[ ph + (1-5p) \right]$$

is necessarily concave (even if $5p$ is not concave) as discussed in class. Thus, the Legendre transform cannot recover $D_{\text{bifractal}}(h)$.

Notice that $\overline{D}(h)$ calculated in part (a) is actually the concave envelope of $D_{\text{bifractal}}(h)$, i.e., the smallest concave function such that $\overline{D}(h) \geq D_{\text{bifractal}}(h)$ for all $h$. 

The Riemann example of an integrable function discontinuous on a dense set

Plot of the discontinuous Khokhlov sawtooth shock solution of inviscid Burgers
Above we have plotted Riemann's function using his series defining truncated at \( n = 100 \). Thus, the errors are of the order of \( 10^{-4} \), quite small. For comparison, we have plotted the Khablov shock solution of inviscid Burgers. The two functions have the same scaling exponents

\[ 5p = \min\{p, 1\} \]

but the first is multifractal and the second bi-fractal! Clearly, Riemann's function is much rougher, with a dense set of discontinuities "shocks" distributed over the unit interval.

(c) Clearly, the example in (c) shows that two functions with distinct multifractal spectra may have the same scaling exponents \( 5p \).

Thus, the inversion

\[ 5p \rightarrow D(h) \]

is not unique!

The Parisi-Frisch inversion formula using the Legendre transform

\[ D_{PF}(h) = \inf_{p} \left\{ \frac{\xi}{p} + (d - 5p) \right\} \]

is necessarily a concave function. Thus, it can only recover the true multifractal spectrum \( D(h) \) of a function \( f \) if \( D(h) \) is concave. In general, \( D_{PF}(h) \) will give the concave envelope \( \overline{D}(h) \) of the true spectrum.
Problem 4(a) We shall again calculate

\[ \kappa(h) = \sup_p \frac{\delta_p}{\ell_p - \phi(h)} \]

for the three models, from which \( D(h) = 3 - \kappa(h) \) can be easily obtained.

Lognormal model (K62): In Homework #8 we found that

\[ h = \frac{d\delta_p}{dp} = \frac{1}{3} - \frac{\mu}{18} (2p - 3) \]

which is easily solved to give

\[ p = \frac{3}{2} + \frac{9}{\mu} \left( \frac{1}{3} - h \right). \]

Since \( \delta_p = \frac{p}{3} \left[ 1 - \frac{\mu}{2} \left( \frac{p}{3} - 1 \right) \right] \) and \( \frac{\delta_p}{\ell_p} = \frac{1}{2} + \frac{3}{\mu} \left( \frac{1}{3} - h \right) \), we get

\[ \kappa(h) = \frac{p}{3} \left[ 1 - \frac{\mu}{2} \left( \frac{p}{3} - 1 \right) \right] - \frac{p}{3} \cdot (3h) \]

\[ = \frac{p}{3} \left[ 1 - \frac{\mu}{2} \left( -\frac{1}{2} + \frac{3}{\mu} \left( \frac{1}{3} - h \right) \right) - 3h \right] \]

\[ = \frac{p}{3} \left[ \frac{1}{2} + \frac{\mu}{4} - \frac{3h}{2} \right] \]

\[ = \left[ \frac{1}{2} + \frac{3}{\mu} \left( \frac{1}{3} - h \right) \right] \left[ \frac{1}{2} + \frac{\mu}{4} - \frac{3h}{2} \right] \]

\[ = \frac{1}{24} \left[ 1 + \frac{\mu}{2} - 3h \right]^2 \]
**log-Poisson (SL):** In Homework #8 we found that

\[ h = \frac{dS_p}{dP} = \frac{1}{q} + \left( \frac{2}{3} \right)^{\frac{p}{3} + 1} \ln \left( \frac{3}{2} \right), \]

which can be inverted to give

\[ \left( \frac{2}{3} \right)^{p/3} = \frac{h - \frac{1}{q}}{\frac{2}{3} \ln \left( \frac{3}{2} \right)} \]

or

\[ p = \frac{3}{\ln \left( \frac{2}{3} \right)} \ln \left( \frac{h - \frac{1}{q}}{\frac{2}{3} \ln \left( \frac{3}{2} \right)} \right) = \frac{3}{\ln \left( \frac{3}{2} \right)} \ln \left( \frac{\frac{2}{3} \ln \left( \frac{3}{2} \right)}{h - \frac{1}{q}} \right). \]

Thus

\[ \kappa(h) = 5p - p^1 \]

\[ = \frac{p}{q} + \frac{2}{\ln \left( \frac{2}{3} \right)} \left[ 1 - \left( \frac{2}{3} \right)^{p/3} \right] - p \left[ \frac{1}{q} + \left( \frac{2}{3} \right)^{p/3} + 1 \ln \left( \frac{3}{2} \right) \right] \]

\[ = 2 - 2 \left( \frac{2}{3} \right)^{p/3} - p \left( \frac{2}{3} \right)^{p/3} \ln \left( \frac{3}{2} \right) \]

\[ = 2 - \left( 2 + \frac{2}{3} \ln \left( \frac{3}{2} \right) \right) \left( \frac{2}{3} \right)^{p/3} \]

\[ = 2 - 2 \left[ 1 + \ln \left( \frac{\frac{2}{3} \ln \left( \frac{3}{2} \right)}{h - \frac{1}{q}} \right) \right] \cdot \frac{h - \frac{1}{q}}{\frac{2}{3} \ln \left( \frac{3}{2} \right)} \]

\[ = 2 - 3 \left[ 1 + \ln \left( \frac{\frac{2}{3} \ln \left( \frac{3}{2} \right)}{h - \frac{1}{q}} \right) \right] \cdot \frac{h - \frac{1}{q}}{\ln \left( \frac{3}{2} \right)}. \]
mean-field (MF): In homework #8 we found that

\[ h = \frac{d\xi}{dp} = \frac{ab}{(b+cp)^2} \]

which is easily inverted to give

\[ p = \frac{1}{c} \left( \frac{ab}{h} \right)^{1/2} - \left( \frac{b}{c} \right) \]

Then,

\[ k(h) = 5p - ph \]

\[ = \frac{ap}{b+cp} - ph = p \left[ \frac{a}{b+cp} - h \right] \]

and using \( b+cp = \left( \frac{ab}{h} \right)^{1/2} \) and the previous result for \( p \) gives

\[ k(h) = \left[ \frac{1}{c} \left( \frac{ab}{h} \right)^{1/2} - \left( \frac{b}{c} \right) \right] \left( \frac{a}{b} \right)^{1/2} h^{1/2} - h \]

\[ = \frac{b}{c} \frac{1}{h^{1/2}} \left[ \left( \frac{a}{b} \right)^{1/2} - h^{1/2} \right] \cdot h^{1/2} \left[ \left( \frac{a}{b} \right)^{1/2} - h^{1/2} \right] \]

\[ = \frac{b}{c} \left[ \left( \frac{a}{b} \right)^{1/2} - h^{1/2} \right]^2 \]

(b) On the next page we plot the three model results for \( D(h) = 3 - k(h) \). They agree quite well over a narrow range of \( h \), roughly \( h \in [0.25, 0.4] \) and disagree substantially outside. It is interesting that this range of \( h \)-values corresponds to \( p \)-values in the range \( p \in [-1.5] \), which is the most accurately known from experiments and simulations.
Theoretical Predictions for the Multifractal Spectrum of Longitudinal Velocity Increments

fractal dimension $D$

Hoelder exponent $h$
Problem 5. (a) By definition,

\[ u^{(n)}(x) = u_n(x) - u_{n-1}(x) \]

\[ = \int d^4r \left[ G_n(r) - G_{n-1}(r) \right] u(x+r) \]

\[ = \int d^4r \left[ G_n(r) - G_{n-1}(r) \right] [u(x+r) - u(x)] \]

Since \( \int d^4r \left[ G_n(r) - G_{n-1}(r) \right] = 1 - 1 = 0 \), if \( G \) is reflection symmetric, then we can change variables in the integral from \( r \to -r \), and obtain

\[ u^{(n)}(x) = \int d^4r \left[ G_n(r) - G_{n-1}(r) \right] [u(x-r) - u(x)] \]

Averaging the two expressions gives

\[ u^{(n)}(x) = \frac{1}{2} \int d^4r \left[ G_n(r) - G_{n-1}(r) \right] [u(x+r) + u(x-r) - 2u(x)] \]

Now, if \( u \) is Hölder continuous at \( x \) with exponent \( 0 < \alpha < 2 \), then (and only then)

\[ \delta^2 u(r; x) = u(x+r) + u(x-r) - 2u(x) = O(|r|^{\alpha}) \]

Hence,

\[ |u^{(n)}(x)| \leq \frac{1}{2} \int d^4r \left[ G_n(r) + G_{n-1}(r) \right] |\delta^2 u(r; x)| \]

\[ \leq \frac{1}{2} \int d^4r \left[ G_n(r) + G_{n-1}(r) \right] \text{const.} |r|^{\alpha} = C n^{\alpha} \]

with \( C = \frac{1}{2} \text{(const.)} \int d^4p \left[ G(p) + \frac{1}{2d} G \left( \frac{p}{2} \right) \right] |p|^{\alpha} \), a finite constant,
If $h \geq 2$, then it need no longer be true that $u^{(n)}(x) = O(r^h/n)$ because

$$\delta^2 u(r; x) = O(1/r^h)$$

is also no longer true! Instead, higher-order differencing is required to detect the Hölder exponent. For example, if $2 \leq h < 4$

$$\delta^4 u(r; x) = u(x+2r) - 4u(x+r) + 6u(x) - 4u(x-r) + u(x-2r)$$

$$= O(1/r^h).$$

In that case,

$$u^{(n-1)}(x) - 4u^{(n-2)}(x) = \frac{1}{2} \int_0^r \left[ G_{a_n}(s) - G_{a_n-(s)} \right] \delta^2 u(r; x)$$

$$= O(r^{-h}).$$

(b) If the signal $u(x)$ is monofractal with exponent $1 < h < 2$, then its derivative $u'(x)$ is monofractal with exponent $0 < h' < 1$, $h' = h - 1$. At an arbitrary point

$$\delta u(r; x) = ru'(x) + O(r^h).$$

The first term $O(r)$ dominates at small $r$ unless $u'(x)$ happens to vanish. This occurs on a set of dimension $1-h' = 2-h$, or codimension $h' = h-1$. Thus,

$$\delta u(r; x) \sim r \quad \text{probability } O(r^0)$$

$$\delta u(r; x) \sim r^h \quad \text{probability } O(r^{h-1}).$$

Using instead the inverse function (exit statistic), we get
\[ r(\delta u) \sim \delta u \quad \text{probability } O(\delta u^0) \]
\[ r(\delta u) \sim (\delta u)^{\nu/n} \quad \text{probability } O(\frac{\delta u}{n}) \]

By the Parisi-Frisch steepest descent argument,

\[ T_p(\delta u) = \left< \left[ r(\delta u) \right]^p \right> \sim (\delta u)^{\kappa_p} \]

with

\[ \kappa_p = \inf_n \left\{ p \cdot 1 + 0, \frac{1}{p} \cdot \frac{1}{n} + \frac{h-1}{n} \right\} \]

\[ = \inf_n \left\{ p, \frac{p}{n} + 1 - \frac{1}{n} \right\} \]

(c) The "bifractality" observed in (b) for a multifractal function is an artifact of using an inappropriate measure of the Hölder singularity $h > 1$. To detect such a singularity $1 < h < 2$, a second-difference

\[ \delta^2 u(\xi; x) = u(x+r) + u(x-r) - 2u(x) \]

must be used:

\[ \delta^2 u(\xi; x) \sim r^h, \text{ for all } x. \]

In that case, the corresponding exit statistic $r^2(\delta u)$ satisfies

\[ r^2(\delta u; x) \sim (\delta u)^{\nu/n} \text{ for all } x, \]

It then follows that

\[ T_p(\delta u) = \left< \left[ r^2(\delta u) \right]^p \right> \sim (\delta u)^{p/h} \]

for all $p > 0$. 
(d) The argument in (c) may be generalized to an arbitrary multifractal spectrum, as long as its dimension spectrum satisfies

\[ \text{supp } D = \{ h : D(h) > -\infty \} \subseteq [0,2] \, . \]

In that case,

\[ r^{(2)}(\delta u, x) \sim (\delta u)^{1/h} \quad \forall x \in g(h) \]

which occurs with a probability

\[ \text{Prob}( r^{(2)}(\delta u) \sim (\delta u)^{1/h} ) \sim \delta u \]

Thus,

\[ \left< r^{(2)}(\delta u) \right>^p \sim \int \mu(\text{d}h) \, \delta u \]

\[ \sim (\delta u)^{X_p^{(2)}} \quad \text{for } \delta u \ll \text{urms} \]

with

\[ X_p^{(2)} = \inf \left\{ \frac{p + (d - D(h))}{h} \right\} \]

or

\[ X_p^{(c)} = \inf \left\{ \frac{p + \kappa(h)}{h} \right\} \]

in terms of the codimension spectrum \( \kappa(h) = d - D(h) \).

This formula may be inverted to give \( \kappa(h) \) from \( X_p^{(c)} \), via

\[ \text{cont'd} \]
\[ \kappa(h) = \sup_p \frac{\varepsilon^{(e)}_p}{p} \]  \[ \therefore \] \[ \text{One way to derive this inverse formula is to rewrite the formula for } \varepsilon^{(e)}_p \text{ as} \]

\[ \varepsilon^{(e)}_p = \inf_{\hat{\nu}} \varepsilon_0 \hat{\nu} + \hat{\kappa}(h) \]

with

\[ \hat{\nu} = \frac{1}{h}, \quad \hat{\kappa}(\hat{\nu}) = \kappa(h)/h, \]

It then follows from the standard multifractal inverse formula that

\[ \hat{\kappa}(\hat{\nu}) = \sup_p \frac{\varepsilon^{(e)}_p}{p} \]

\[ \therefore \]

\[ \frac{1}{h} \kappa(h) = \sup_p \frac{\varepsilon^{(e)}_p}{p} - \frac{p}{h} \]

\[ \therefore \text{, multiplying through by } h, \quad (\ast) \]

Thus, using the formula $(\ast)$, one may obtain the dimension spectrum for the multifractal function with $h \in [0, 2]$, including the values $h > 1$. However, this is not true of the quantity considered by Biferale et al.

\[ \tau(\delta u) = \inf \bar{r} : \delta u_{\bar{r}}(r) = \delta u \]

which is defined in terms of the first difference $\delta u_{\bar{r}}(r)$, because

\[ \delta u_{\bar{r}}(r) \sim r \cdot \delta u + O(r^h) \]

at points with $h > 1$ and this is dominated by the first
term which is $O(r)$. Thus, all points with $h > 1$
contribute in the same way to

$$T_p(\delta u) = \langle [r(\delta u)]^p \rangle \sim (\delta u)^{xp}.$$ 

The exponent $xp$ therefore contains no information about
the multifractal spectrum $D(h)$ with $h > 1$.

**Remark:** A biproduct of the above discussion is that
the inverse structure functions contain the same information
for $p > 0$ that the ordinary (direct) structure functions
certain for $p < 0$. To show this, assume, for simplicity,
that $\text{supp} D \subset [0,1]$ so that the quantity $T_p(\delta u)$ suffices.
In that case, we may write, just as above, that

$$xp = \inf_r \left\{ \frac{p + \zeta(h)}{h} \right\}$$

$$= \inf_r \left\{ p \frac{\hat{h}}{\hat{h}} + \zeta(\hat{h}) \right\}$$

with $\hat{h} = 1/h$, $\zeta(\hat{h}) = \zeta(h)/h$. The usual multifractal formalism
then implies that

$$\frac{dX_p}{dp} = \hat{h}_p = \frac{1}{h_p}$$

where $h_p$ and $\hat{h}_p$ are the points at which the above two infimums
are achieved. Since $X_p$ is concave (by Hölder inequality),
$\hat{h}_p = dX_p/dp$ must be decreasing in $p$. But in that case,
$\hat{h}_p$ is increasing in $p$. Thus, $p > 0$ corresponds to $h_p$ greater
than the most probable value $h_0$. This is the main reason

**to consider inverse structure functions**