1. This problem explores an alternative definition of the local Hölder exponent based on the quantity \( \delta u(\ell) = \sup_{r<\ell} |\delta u(r)|. \) This definition is closely related to the “maximum modulus methods” used to determine Hölder scaling exponents experimentally. Assume that the limit

\[
h_* = \lim_{\ell \to 0} \frac{\ln \delta u(\ell)}{\ln(\ell/L)}
\]

exists (possibly = +\( \infty \)).

(a) If \( u \) is continuous, show that for \( \ell \) small enough and \( r < \ell \)

\[
\frac{\ln |\delta u(r)|}{\ln(r/L)} < \frac{\ln |\delta u(\ell)|}{\ln(\ell/L)}.
\]

Use this result and the identity

\[
\frac{\ln \delta u(\ell)}{\ln(\ell/L)} = \inf_{r<\ell} \frac{\ln |\delta u(r)|}{\ln(\ell/L)}
\]

for \( \ell < L \) to conclude that \( h \leq h_* \), where \( h \) is the maximal Hölder exponent.

(b) From the definition of \( h_* \), show that for every \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that for all \( r < \ell < \delta \)

\[
|\delta u(r)| \leq \left( \frac{\ell}{L} \right)^{h_* - \epsilon}.
\]

Conclude that \( u \) is Hölder continuous with exponent \( h_* - \epsilon \) and, thus, that \( h \geq h_* \).

2. This problem explores some other notions of “fractal dimension”. A set \( A \) is said to be self-similar if there exists an integer \( r > 1 \) such that

\[
rA = \bigcup_{i=1}^{c} A_i.
\]

In other words, \( A \) under rescaling by \( r \) is a disjoint union of \( c \) sets \( A_i, \ i = 1, ..., c \) that are each obtained from \( A \) by a rigid rotation and translation. The similarity dimension of a self-similar set \( A \) is defined as

\[
D_S(A) = \frac{\log c}{\log r}.
\]

See K. J. Falconer, The Geometry of Fractal Sets (Cambridge University Press, 1985), Chapter 8, where it is shown that the similarity dimension equals both the box-counting dimension and the Hausdorff dimension for any self-similar set.
(a) Show that the two-thirds Cantor set $K$ is self-similar with $r = 3$ and $c = 2$, so that $D_S(K) = \log 2 / \log 3$.

(b) The Koch curve $C$ is defined by repeating the operation on segments of subtracting the middle third segment and adding two more segments of equal length:

The curve obtained after five iterations of this operation appears as follows:

Show that the Koch curve is self-similar and calculate its similarity dimension.

(c) Another notion of dimensionality is the Hausdorff dimension $D_H(A)$ of a set $A$. This quantity is defined in terms of the “$d$-dimensional Hausdorff (outer) measure” $\mathcal{H}^d(A)$, which generalizes the concept of length ($d = 1$), area ($d = 2$) and volume ($d = 3$) of the set $A$ to a noninteger “$d$-dimensional volume.” For every set $A$, there is a critical value $d_c$ such that $\mathcal{H}^d(A) = +\infty$ for $d < d_c$ and $\mathcal{H}^d(A) = 0$ for $d > d_c$. This critical value $d_c$ is defined as the Hausdorff dimension $D_H(A)$ of the set $A$. See Falconer (1985) for more details.

Show that $n$th stage in the construction of the Cantor set $K_n$ has length $L(K_n) = (2/3)^n$, while the $n$th stage of the Koch curve $C_n$ has length $L(C_n) = (4/3)^n$. Conclude that $L(K) = 0$ and $L(C) = +\infty$. What does this imply about $D_H(K)$ and $D_H(C)$?

3. This problem more carefully investigates the “bifractal” scaling exponents

$$\zeta_p = \min\{p, 1\} = \begin{cases} p & p < 1 \\ 1 & p \geq 1 \end{cases}$$

(a) Show that the Legendre transform $\overline{D}(h) = \inf_p [ph + (1 - \zeta_p)]$ of this set of exponents does not give a “bifractal” result but instead a nontrivial multifractal spectrum

$$\overline{D}(h) = \begin{cases} h & h \in [0, 1] \\ -\infty & \text{otherwise} \end{cases}$$
(b) Explain why the Legendre transform cannot give the “bifractal” spectrum

\[ D_{\text{bifractal}}(h) = \begin{cases} 
0 & h = 0 \\
1 & h = 1 \\
-\infty & \text{otherwise}
\end{cases} \]

which is the true dimension spectrum for Burgers turbulence.

(c) In his Habilitationsschrift of 1854, Bernard Riemann introduced the concept of integration now named after him. In that work, Riemann defined several functions that were too irregular to be Cauchy integrable but nevertheless “Riemann integrable.” One of these examples is the function with period 1 defined by:

\[ R(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2} \]

where \((x) = x - k\) if \(|x - k| < 1/2\) and \((x) = 0\) if \(x = k + 1/2\), for all \(k \in \mathbb{Z}\). This function has the same scaling exponents \(\zeta_p = \min\{p, 1\}\) for \(p \geq 0\) as does Burgers turbulence, but has the multifractal dimension spectrum \(D(h)\) calculated in part(a). See S. Jaffard, J. Fourier Analysis & Applications 3 1-22 (1997). Plot Riemann’s function using the above series truncated at large \(n\) and compare with Khokhlov’s sawtooth solution of inviscid Burgers equation.

(d) In light of the example in (c), can the multifractal spectrum be uniquely determined from the scaling exponents \(\zeta_p\)? State a general necessary condition on the spectrum \(D(h)\) for it to be recoverable from the Parisi-Frisch formula.

4. In Homework #8 of Turbulence I we considered several theoretical models for the scaling exponents of the longitudinal velocity structure functions. These were the lognormal model of A. N. Kolmogorov, J. Fluid Mech. 13 82 (1962)

(K62) \[ \zeta_p = \frac{p}{3} - \frac{\mu}{18} p(p - 3) \]

with \(\mu = 0.25\), the log-Poisson model of Z.-S. She and E. Lévêque, Phys. Rev. Lett. 72 336 (1994)

(SL) \[ \zeta_p = \frac{p}{9} + 2 \left[ 1 - \left( \frac{2}{3} \right)^{p/3} \right] \]

and the “mean-field theory” of V. Yakhot, Phys. Rev. E 63 026307 (2001)

(MF) \[ \zeta_p = \frac{ap}{b + cp} \]

with \(a = 0.185, b = 0.473\) and \(c = 0.0275\).
(a) For each of these theoretical models for $\zeta_p$ calculate the dimension spectrum $D(h)$ that follows from the Parisi-Frisch formula. **Hint:** Invert the result for $h(p)$ from Homework #8 to find $p(h)$ and substitute into the formula $D(h) = p(h)h + (3 - \zeta(p(h)))$.

(b) Plot the spectra $D(h)$ from part (a) for the three models, in particular for the range $0 < h < 1$. Comment on the similarities and differences.

5. This problem considers issues associated with Hölder exponents $h > 1$, which may appear in some applications (including, possibly, three-dimensional turbulence).

(a) If the kernel $G$ is reflection-symmetric, so that $G(-r) = G(r)$, then show that the band-pass filtered field defined in the class notes equals

$$u^{[n]}(x) = \frac{1}{2} \int d\ell [G_{\ell_n}(r) - G_{\ell_{n-1}}(r)][u(x+r) + u(x-r) - 2u(x)].$$

Use this result to show that for such a kernel $u^{[n]}(x) = O(\ell^h)$ if $u$ is Hölder continuous at $x$ with any exponent $0 < h < 2$. Will a similar estimate necessarily hold if $h \geq 2$?

(b) L. Biferale et al., “Inverse Statistics of Smooth Signals: The Case of Two Dimensional Turbulence,” Phys. Rev. Lett. 87 124501 (2001) apply “inverse structure functions” as an alternative to structure functions of negative order $p$. In this approach, one defines $r(\delta v)$ for $\delta v > 0$ as the “first-exit statistic,” i.e. the smallest value of $r > 0$ such that $|\delta u_L(r)| = \delta v$. The (longitudinal) inverse structure function of order $p$ is then defined as

$$T_p(\delta v) = \langle [r(\delta v)]^p \rangle.$$

Those authors consider a monofractal velocity $u$ with exponent $1 \leq h < 2$ and show that it has so-called “bifractal scaling” $T_p(\delta v) \sim (\delta v)^{\chi_p}$ when $\delta v \ll u_{rms}$ with

$$\chi_p = \inf \left\{ p, \frac{p}{h} + 1 - \frac{1}{h} \right\}.$$

Explain carefully the above formula for $\chi_p$. **(Hint:** You will need the result for a monofractal function with Hölder exponent $0 < \xi < 1$ that the set of its zero-crossings has dimension $1 - \xi$.)

(c) Is the result in part (b) true “bifractality”? Show that if one defines the “exit-statistic” $r^{(2)}(\delta v)$ as the smallest value of $r > 0$ such that $|\delta^2 u_L(r)| = \delta v$ for the second-difference $\delta^2 u_L(r)$, then whenever $p > 0$

$$T_p^{(2)}(\delta v) \sim (\delta v)^{p/h}, \; \delta v \ll u_{rms}$$

for any monofractal velocity $u$ with $0 < h < 2$. 

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(d) Generalize the result in (c) to any multifractal velocity $u$ with all Hölder exponents in the range $0 < h < 2$ by arguing for the formula

$$
\chi_p^{(2)} = \inf_h \left\{ \frac{p + (d - D(h))}{h} \right\}, \quad p > 0
$$

of the scaling exponent of $T_p^{(2)}(\delta v)$ at $\delta v \ll u_{rms}$. Explain why this quantity, if measured from experimental or DNS data, might reveal a nontrivial multifractal spectrum $D(h)$ for $h > 1$, whereas the quantity considered by Biferale et al. (2001) is unsuitable for this purpose.

*Remark:* It would be very interesting to measure $\chi_p^{(2)}$ in the two-dimensional enstrophy cascade, the same situation considered by Biferale et al. (2001). It should detect nontrivial multifractality associated with smooth, coherent vortex regions of the flow that was missed by those authors!