Final Exam 550.386, May 4, 2007

Do any **five** *of the following six problems.* Circle the number of the five problems that you wish to be graded for credit. Show all your work; answers without supporting work may receive no credit.

The exam is **open book**, and any resource (notes, textbook, webpage) may be used, so long as you quote the information employed and its source in your solution.

I attest that I have completed this exam without unauthorized assistance from any person, materials, or device:

Full name: ____

Signature: ____

(See the Johns Hopkins Handbook *Academic Ethics for Undergraduates*).

1. Below is the array of finite-differences that is produced in applying the Newton algorithm to find the interpolating polynomial for the six points $(x, y) = (1, 7), (2, -4), (3, 3), (4, -2), (5, 11), (6, 4)$:

x	y	$1st$	$2nd$	$3rd$	$4th$	$5th$
1.0000	7.0000	0	0	0	0	0
2.0000	−4.0000	−11.0000	0	0	0	0
3.0000	3.0000	7.0000	9.0000	0	0	0
4.0000	−2.0000	−5.0000	−6.0000	−5.0000	0	0
5.0000	11.0000	13.0000	9.0000	5.0000	2.5000	0
6.0000	4.0000	−7.0000	−10.0000	−6.3333	−2.8333	−1.0667

The last five columns contain the 1st, 2nd, 3rd, 4th and 5th differences, respectively.

(a) Write down the interpolating polynomial $P_4(x)$ for the five points $(x, y) = (1, 7)$, $(2, -4)$, $(3, 3)$, $(4, -2)$, $(5, 11)$.

(b) Write down the interpolating polynomial $P_5(x)$ for all six points $(x, y) = (1, 7)$, $(2, -4)$, $(3, 3)$, $(4, -2)$, $(5, 11)$, and $(6, 4)$. If a seventh point is added with $x = 7$, how will the interpolating polynomial $P_6(x)$ change?

(c) Assume that the points are obtained from a smooth function f as $(x, y) = (x, f(x))$. Write down the error in approximating $f(x)$ at the point x with the polynomial $P_6(x)$, using an expression which involves a suitable finite-difference of f.

(d) Continuing (c), write down a formula for the error in approximating the derivative $f'(x)$ at the point x with the polynomial $P_6'(x)$, using again finite-differences of f.

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2. (a) Consider three times t_{n-1}, $t_n = t_{n-1} + h$, $t_{n+1} = t_n + h$ for some $h > 0$ and three vector values $\mathbf{y}_{n-1}$, $\mathbf{y}_n, \mathbf{y}_{n+1}$. Write down the Lagrange form of the interpolating polynomial $\mathbf{P}_2(t)$ that passes through the points $(t_i, \mathbf{y}_i)$ for $i = n-1, n, n+1$.

(b) Calculate the derivative $\mathbf{P}_2'(t)$.

(c) Use part (b) to construct the 2-point Backward Differentiation Formula (BDF2)

$$\mathbf{P}_2'(t_{n+1}) \doteq \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}).$$

Write this in the standard form

$$\mathbf{y}_{n+1} = \alpha_0 \mathbf{y}_n + \alpha_1 \mathbf{y}_{n-1} + \beta h \mathbf{f}(t_{n+1}, \mathbf{y}_{n+1}).$$

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3. Consider a general 4th-order Runge-Kutta scheme

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{F}(t_n, \mathbf{y}_n; \mathbf{f}, h)$$

with local truncation error $\mathbf{T}_n(\mathbf{y}) = \boldsymbol{\varphi}(t_n)h^5 + O(h^6)$.

(a) If $\mathbf{y}(t)$ is the exact solution of the ODE $\dot{\mathbf{y}} = \mathbf{f}(t, \mathbf{y})$ and if $\mathbf{y}(t; h)$ is the approximate solution with stepsize h, there is an asymptotic error formula of the form

$$\mathbf{y}(t) - \mathbf{y}(t; h) = \boldsymbol{\delta}(t)h^4 + O(h^5)$$

for $h \to 0$. What ODE does the coefficient $\boldsymbol{\delta}(t)$ satisfy?

(b) Use the above asymptotic error formula for approximate solutions $\mathbf{y}(t; h)$ and $\mathbf{y}(t; 2h)$ to derive an error estimate involving only these two approximate solutions. (*Hint:* eliminate the exact solution!)

(c) Apply the error estimate in (b) to approximate the error in calculating $y(1) = e$ by solving $\dot{y} = y$, $y(0) = 1$ with the classical 4th-order Runge-Kutta method for $N = 20$ steps, by comparing with the solution for $N = 10$ steps. Determine the true error and the approximation to the error.

(d) Use part (c) and extrapolation to develop an improved estimate for $y(1) = e$. Determine the error in the extrapolated estimate.

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4. The optimal second-order Runge-Kutta scheme has the form

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\left[\frac{1}{4}\mathbf{k}_1 + \frac{3}{4}\mathbf{k}_2\right]$$

with

$$\mathbf{k}_1 = \mathbf{f}(t_n, \mathbf{y}_n), \quad \mathbf{k}_2 = \mathbf{f}\left(t_n + \frac{2}{3}h, \mathbf{y}_n + \frac{2}{3}h\mathbf{k}_1\right).$$

This method has the smallest error coefficient of all 2nd-order RK algorithms.

(a) Calculate the characteristic equation of this 1-step method and find its root(s).

(b) Is this method weakly stable? Relatively stable? Explain your answers.

(c) Find the stability threshold of this method. Is the scheme A-stable?

5. Consider the following 2-step numerical scheme:

$$\mathbf{y}_{n+1} = \frac{3}{2}\mathbf{y}_{n-1} - \frac{1}{2}\mathbf{y}_n + \frac{5}{2}h\mathbf{f}(t_n, \mathbf{y}_n).$$

(a) Is this method consistent?

(b) Find the characteristic equation of the method and its two roots $r_0(z)$, $r_1(z)$.

(c) Is the method weakly stable? Convergent? When can this method be used?

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6. Consider the following initial-value problem

$$\dot{y} = -e^t y + 1 - e^{-t}, \quad y(0) = 1$$

for the time interval $0 < t < 8$. The exact solution is $y(t) = e^{-t}$.

(a) Consider solving this problem with Heun's method for a fixed time-step h. Using the stability threshold of this method, determine the largest step h and, thus, the smallest number of steps N so that the Heun integration remains absolutely stable for $0 < t < 8$.

(b) Apply the Heun method with $N = 10^4$ and report the results. Explain your observations using part (a).

(c) Repeat (a) for the trapezoidal method and (b) for the trapezoidal method with $N = 100$.

(d) Which of the following MATLAB integrators, `ode45`, `ode113` and `ode15s`, give convergent results for the above initial-value problem in the limit $TOL \to 0$? State which of these methods you would use for this problem and explain why.

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