Midterm 550.386, March 7, 2005.

Do any **three** *of the following four problems.* Circle the number of the three problems that you wish to be graded for credit. Show all your work. Answers without supporting work may receive no credit.

1. According to a standard trigonometric identity, the following two expressions are equal in exact arithmetic:

$$x_1 = \sin(3.0000001) - \sin(2.9999999)$$
$$x_2 = 2\cos(3)\sin(0.0000001).$$

(a) Use `MATLAB` to evaluate x_1 and x_2. How many significant figures do they have in common?

(b) How many significant figures are there in IEEE standard floating point double precision arithmetic? Do the `MATLAB` results for x_1 and x_2 agree to that precision?

(c) Which of the `MATLAB` results do you expect to be most accurate, that for x_1 or for x_2? Explain your answer.

(d) Estimate the relative error in the least accurate result. How does this relate to the result in (a)?

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2. Each of the following two functions has $x = 0$ as its only root:

$$f_1(x) = x^{1/3}$$
$$f_2(x) = x^3.$$

For both of these functions, answer the following questions:

(a) Does the Newton method converge, starting in any neighborhood of $x = 0$? Prove that your answer is correct.

(b) If the Newton method converges, what is the order of convergence? Justify your answer.

(c) Can you modify the Newton method so that it converges faster? If so, explain how you would do it.

3. Consider the function

$$g(x) = \frac{e^{x/2}}{\sqrt{3}}.$$

(a) Prove that there exists an interval $I = (a, b)$ on the real line such that the fixed point iteration $x_{n+1} = g(x_n)$ converges, if $x_0 \in I$.

(b) Determine the fixed point x_* numerically in `MATLAB` to a tolerance $eps = 10^{-15}$.

(c) Determine the asymptotic order of convergence. If the order is linear, calculate the rate constant.

(d) Use the result in (c) to estimate theoretically the number of iterations required to converge with tolerance $eps = 10^{-15}$, starting from $x_0 = 2$. Does this compare well with the actual number of iterations required?

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4. Consider the following polynomial of degree four:

$$p(x) = 2x^4 - 33x^3 + 29x^2 + 66x - 62.$$

(a) Use Descartes' rule of signs to determine the possible number of positive real roots and negative real roots.

(b) Determine all of the real roots numerically in `MATLAB` as precisely as possible. You may use any method you like, but explain your approach. If you use a standard `MATLAB` routine, you must explain briefly what algorithm it uses.

(c) Suppose that the coefficient of the x^4 term is changed from 2 to 2.01. Estimate theoretically the effect that this has on the value of the smallest positive root x_1.

(d) Numerically determine the smallest positive root of the modified polynomial in (c) using `MATLAB` and compare with your theoretical estimate.