4. (a) For the problem \( \dot{x} = x, \ x(0) = 1 \) with exact solution \( X(t) = e^t \), give the explicit solution \( x_n \) for Euler’s approximation to the equation. Use this explicit formula to show that for \( t_n = 1 \), \( X(1) - x_n = (h/2)e \) as \( h \to 0 \).

(b) Show that the same error estimate is obtained from the general asymptotic error formula for Euler’s method:

\[
X(t_n) - x_n = \delta(t_n)h + O(h^2),
\]

where \( \delta(t) \) is the solution of the linear initial-value problem

\[
\dot{\delta}(t) = f_x(t, X(t))\delta(t) + \frac{1}{2} \ddot{X}(t), \quad \delta(t_0) = \delta_0.
\]

Solution: a) The explicit solution \( x_n \) for Euler’s approximation can be found as follows:

with \( h = \frac{1}{n} \) we have

\[
\begin{align*}
x_0 &= 1 \\
x_1 &= 1 + hx_0 = 1 + h \\
x_2 &= x_1 + hx_1 = (1 + h)x_1 = (1 + h)^2 \\
&\quad\vdots \\
x_n &= x_{n-1} + hx_{n-1} = (1 + h)x_{n-1} = (1 + h)^n
\end{align*}
\]

Thus, \( x_n = (1 + h)^{1/h} \) since \( n = 1/h \). We are interested in the limit of \( x_n = (1 + h)^{1/h} \) as \( h \to 0 \). Let us consider

\[
\ln\left((1 + h)^{1/h}\right) = \frac{1}{h} \ln(1 + h) = \frac{1}{h} \left(h - \frac{1}{2} h^2 + O(h^3)\right) = 1 - \frac{1}{2} h + O(h^2)
\]

Therefore,

\[
x_n = (1 + h)^{1/h} = e^{\ln((1 + h)^{1/h})} = e^{1 - \frac{1}{2} h + O(h^2)} = e \times e^{-\frac{1}{2} h + O(h^2)} = e \left(1 - \frac{1}{2} h + O(h^2)\right)
\]

and finally,

\[
X(1) - x_n = e - (1 + h)^{1/h} = e\left(\frac{1}{2} h + O(h^2)\right) = \frac{h}{2} e + O(h^2).
\]

as \( h \to 0 \).

b) Specific to the problem here,

\[
\dot{\delta}(t) = f_x(t, X(t))\delta(t) + \frac{1}{2} \ddot{X}(t) = \delta(t) + \frac{1}{2} e^t
\]

This equation can be explicitly integrated (e.g. multiply both sides by \( e^{-t} \) to obtain \( \frac{d}{dt} e^{-t} \delta(t) = \frac{1}{2} \)) yielding the solution:

\[
\delta(t) = \frac{1}{2} (t + 2\delta_0)e^t = \frac{1}{2} te^t
\]

where we used in the last step the fact that \( \delta(0) = 0 \). We thus have

\[
X(1) - x_n = \delta(1)h + O(h^2) = \frac{1}{2} eh + O(h^2)
\]

which agrees with the calculation from part a).