1. (a) Use the bisection method to find the roots of
\[ f(x) = e^x - 1 - x^3 \]
with tolerance \( \epsilon = 10^{-15} \). Make certain that you find all of the roots of the given function. Record the final endpoints of the bracketing interval, the achieved tolerance, the number of iterations, and the computational time for each root. (You may use the prepared matlab script \texttt{bisect.m}, if you wish.)

(b) Repeat part (a) using instead the Newton-Fourier method discussed in the course lectures. You will need to write your own matlab script (e.g. by modifying \texttt{newton.m}). For each root use the same initial interval \([a, b]\) as you did for the bisection method in part (a). Note that the function may not satisfy the criteria for application of the Newton-Fourier method, for each of the roots. In that case, you will need to transform the function as discussed in class.

(c) Compare the results of the bisection and Newton-Fourier methods (final endpoints of the bracketing interval, achieved tolerance, number of iterations, and computational time). Which method is preferable and why?

2. (a) Use Newton’s method to find the positive root of smallest magnitude of
\[ f(x) = \exp(\tanh(x)) - 2, \]
again with tolerance \( \epsilon = 10^{-15} \) and \( x_0 = 0.8 \). Record the final approximate root, the achieved tolerance, the number of iterations, and the computational time.

(b) Repeat part (a) using the secant method. Use the same initial guess \( x_0 = 0.8 \) as you did for Newton’s method in (a) and make one Newton iteration to generate \( x_1 \).

(c) Repeat part (a) using the method of inverse quadratic interpolation (IQI). Use the same initial \( x_0, x_1 \) as you did for the secant method in (b), and make one secant iteration to generate \( x_2 \).

(d) Compare the results of the three methods (final endpoints of the bracketing interval, achieved tolerance, number of iterations, and computational time). Which method is preferable and why?

3. Derive the error formula for the secant method:
\[ x_* - x_{n+1} = -(x_* - x_n)(x_* - x_{n-1}) \frac{f[x_{n-1}, x_n, x_*]}{f[x_{n-1}, x_n]}, \]
4. A sequence \( \{x_n\} \) is said to converge superlinearly to \( x^* \) if

\[
|x^* - x_{n+1}| \leq c_n |x^* - x_n|, \quad n \geq 0
\]

with \( c_n \to 0 \) as \( n \to \infty \). Show that in this case,

\[
\lim_{n \to \infty} \frac{x^* - x_n}{x_{n+1} - x_n} = 1
\]

Thus \( |x^* - x_n| \leq |x_{n+1} - x_n| \) is increasingly valid as \( n \to \infty \).

5. Newton’s method for finding a root \( x^* \) of \( f(x) = 0 \) sometimes requires the initial guess \( x_0 \) to be quite close to \( x^* \) in order to obtain convergence. Verify that this is the case for the root \( x^* = \pi/2 \) of

\[
f(x) = \cos(x) + \sin^2(50x)
\]

Based on the convergence proof, make a rough estimate how small \( |x_0 - x^*| \) must be for iterates to converge to \( x^* \). Check this estimate numerically by slowly increasing the distance of \( x_0 \) from \( x^* \) until the Newton iteration fails to converge.

6. Define an iteration formula by

\[
x_{n+1} = \bar{x}_{n+1} - \frac{f(\bar{x}_{n+1})}{f'(x_n)}, \quad \bar{x}_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.
\]

Show that the order of convergence of \( \{x_n\} \) to \( x^* \) is at least 3. State conditions on the mean times \( \tau_f \) and \( \tau_{f'} \) to evaluate \( f \) and \( f' \), respectively, so that this method converges asymptotically faster than Newton’s method.

7. Given below is a table of iterates from a linearly convergent iteration \( x_{n+1} = g(x_n) \).

Estimate from this data (a) the rate of linear convergence, (b) the error \( x_7 - x^* \) and (c) the fixed point \( x^* \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.5000000</td>
</tr>
<tr>
<td>1</td>
<td>0.5328847</td>
</tr>
<tr>
<td>2</td>
<td>0.5536649</td>
</tr>
<tr>
<td>3</td>
<td>0.5662537</td>
</tr>
<tr>
<td>4</td>
<td>0.5736789</td>
</tr>
<tr>
<td>5</td>
<td>0.5779878</td>
</tr>
<tr>
<td>6</td>
<td>0.5804645</td>
</tr>
<tr>
<td>7</td>
<td>0.5818802</td>
</tr>
</tbody>
</table>