7. Prove the formula for the relative error in a quotient:

\[ \text{Rel} \frac{x_A}{y_A} = \frac{\text{Rel}(x_A) - \text{Rel}(y_A)}{1 + \text{Rel}(y_A)}. \]

Let \( x_A = x_T + \delta x \) and \( y_A = y_T + \delta y \). Then

\[
\begin{align*}
\text{Rel} \left( \frac{x_A}{y_A} \right) &= \frac{x_A / y_A - x_T / y_T}{x_T / y_T} = \frac{x_A y_T - x_T y_A}{x_T y_A} \\
&= \frac{y_T \delta x - x_T \delta y}{x_T (y_T + \delta y)} = \frac{\delta x / x_T - \delta y / y_T}{1 + \delta y / y_T} \\
&= \frac{\text{Rel}(x_A) - \text{Rel}(y_A)}{1 + \text{Rel}(y_A)}
\end{align*}
\]

8. Give exact ways of avoiding loss-of-significance errors in the following functions and evaluate them to double precision for the given input \( x \):

(a) \( f(x) = \cos^2(x) - \cos(2x), \quad x = 10^{-8} \)

(b) \( f(x) = \sqrt{1 + x^4} - x^2, \quad x = 10^4 \)

We want to express \( f(x) \) in a way that avoids subtracting potentially nearly equal terms.

(a) Using the double-angle formula, we obtain

\[
f(x) = \cos^2(x) - \cos(2x) = \cos^2(x) - (\cos^2(x) - \sin^2(x)) = \sin^2(x).
\]

By Taylor’s theorem,

\[
\sin(x) = x - \frac{x^3}{6} + O(x^5) = x(1 - \frac{x^2}{6} + O(x^4)),
\]

which gives us

\[
\sin^2(x) = x^2(1 - \frac{x^2}{3} + O(x^4)).
\]

Therefore,

\[
f(10^{-8}) = 10^{-16}(1 - \frac{10^{-16}}{3} + O(10^{-32})) \approx 10^{-16}
\]

where the term in the parentheses is a distance less than the unit round \( u = \varepsilon/2 \) from 1.

(b) Rewrite \( f(x) \) using the binomial series as follows:

\[
f(x) = x^2 \left( \sqrt{1 + \frac{1}{x^4}} - 1 \right) = x^2 \left( \left(1 + \frac{1}{2x^4} - \frac{1}{8x^8} + O \left( \frac{1}{x^{12}} \right) \right) - 1 \right)
\]

\[
= \frac{1}{2x^2} \left( 1 - \frac{1}{4x^4} + O \left( \frac{1}{x^8} \right) \right), \quad \text{and therefore}
\]

\[
f(10^4) = 0.5 \times 10^{-8} \left( 1 - \frac{10^{-16}}{4} + O(10^{-32}) \right) \approx 5 \times 10^{-9}
\]
9. Assume that \( x_A = 0.534 \) has three significant digits with respect to \( x_T \). Bound from below and from above the relative error and the absolute error in \( x_A \). For \( f(x) = e^x \), obtain possible intervals for the error and relative error in \( f(x_A) \) with respect to \( f(x_T) \). Also, use the rough estimate

\[
\text{Rel}(f(x_A)) \approx \frac{f'(x_A)}{f(x_A)} \text{Err}(x_A)
\]

in order to approximate \( \text{Rel}(e^{x_A}) \) and compare this with your exact bounds.

Since \( x_a \) has three significant digits with respect to \( x_T \), we can say that

\[-5 \times 10^{-3} \leq \frac{x_A}{x_T} - 1 \leq 5 \times 10^{-3}, \]

which bounds \( \text{Rel}(x_A) \). Then

\[
0.995 \leq \frac{x_A}{x_T} \leq 1.005 \implies \frac{x_A}{1.005} \leq x_T \leq \frac{x_A}{0.995}, \text{ so we have}
\]

\[
x_A(1 - \frac{1}{0.995}) \leq x_A - x_T \leq x_A(1 - \frac{1}{1.005}), \text{ i.e.}
\]

\[-0.002656716418 \leq \text{Err}(x_A) \leq 0.002656716418\]

Then

\[
\text{Err}(f(x_A)) = e^{x_A} - e^{x_T} = e^{x_A} (1 - e^{x_T - x_A}) = e^{x_A} \left( 1 - e^{-\text{Err}(x_A)} \right)
\]

Now, since \( f(u) = 1 - e^{-u} \) is increasing in \( u \), we can say

\[
e^{x_A} \left( 1 - e^{-\text{Err}(x_A)_{\text{min}}} \right) \leq \text{Err}(f(x_A)) \leq e^{x_A} \left( 1 - e^{-\text{Err}(x_A)_{\text{max}}} \right)
\]

\[
e^{\text{Err}(x_A)_{\text{max}}} \left( 1 - e^{-\text{Err}(x_A)_{\text{min}}} \right) \leq \text{Rel}(f(x_A)) \leq e^{\text{Err}(x_A)_{\text{max}}} \left( 1 - e^{-\text{Err}(x_A)_{\text{max}}} \right)
\]

Plugging in the respective min and max values of \( \text{Err}(x_A) \) gives us

\[-0.002667355391 \leq \text{Rel}(x_A) \leq 0.002660248616\]

Using the given approximation \( \text{Rel}(f(x_A)) \approx \frac{f'(x_A)}{f(x_A)} \text{Err}(x_A) \) and noting that \( f'(x) = e^x = f(x) \), we have \( \text{Rel}(f(x_A)) \approx \text{Err}(x_A) \). Comparing this with the obtained bounds, it seems to be a fairly good approximation since the obtained bounds on \( \text{Err}(x_A) \) and \( \text{Rel}(x_A) \) are quite close to one another.