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Rootfinding for a transcendental equation without a first guess: Polynomialization of Kepler's equation through Chebyshev polynomial expansion of the sine ☆

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Abstract

The Kepler equation for the parameters of an elliptical orbit, $E - \varepsilon \sin(E) = M$, is reduced from a transcendental to a polynomial equation by expanding the sine as a series of Chebyshev polynomials. The single real root is found by applying standard polynomial rootfinders and accepting only the polynomial root that lies on the interval predicted by rigorous theoretical bounds. A complete Matlab implementation is given in full because it requires just seven lines. For a polynomial of degree fifteen, the maximum absolute error over the whole range $\varepsilon \in [0, 1]$ and all M is only 4×10^{-10}. Other transcendental equations can similarly be reduced to polynomial equations through Chebyshev expansions.

Keywords: Rootfinding; Chebyshev polynomials; Kepler equation

1. Introduction

The Kepler equation for determining the eccentric anomaly in celestial mechanics is

$$E - \varepsilon \sin(E) - M = 0 \tag{1}$$

where M is the so-called "mean anomaly", ε is the eccentricity of an elliptical orbit, and the unknown E is the "eccentric anomaly", sometimes denoted by ψ. The same equation also arises in the parallax problem as shown by Habash al-Hasid in the ninth century. The method of characteristics yields an exact but implicit solution for the one-dimensional advection equation in hydrodynamics: the eccentricity is the time coordinate and the parabolic orbit case ($\varepsilon = 1$) is the instant of wave-breaking [14,2,3].

Many luminaries have attacked this equation including Newton (who invented Newton's iteration to solve it), Lagrange (who developed a power series in ε), Bessel (who invented Bessel functions to obtain an infinite series solution) and Cauchy (who proved Lagrange's series converged only for $\varepsilon < 0.66$). Indeed, Colwell's history of the

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Kepler equation [10] contains 465 references and it seems likely, allowing for more recent publications and obscure works, that at least half a thousand articles have been devoted to Kepler's equation.

Bessel derived the exact analytical solution

$$E = M + \sum_{m=1}^{\infty} J_m(m\varepsilon)\sin(mM), \quad \varepsilon \leqslant 1,\ M \in [-\pi, \pi]. \tag{2}$$

Charles and Tatum [9] and Thorlund-Petersen [18] showed that Newton's iteration will converge for all $\varepsilon \in [0, 1]$, $M \in [-\pi, \pi]$ from the initial guess $E_0 = \mathrm{sign}(M)\pi$.

There would seem to be nothing further to say. Nonetheless, in this note we show that this transcendental equation is, in a practical sense, merely a polynomial equation. Robust polynomial root-finders are built into Matlab™ and similar systems. These root-finders can be exploited by expanding the sine function in Kepler's equation as a Chebyshev polynomial series [13]:

$$\sin(\pi x) = 2\sum_{k=0}^{\infty}(-1)^k J_{2k+1}(\pi)T_{2k+1}(x), \quad x \in [-1, 1]. \tag{3}$$

Truncating this series, replacing the Chebyshev polynomials by their explicit forms and collecting powers of x converts the result into an ordinary polynomial in the variable $x = E/\pi$. One can then invoke the usual polynomial rootfinders to solve Kepler's equation.

One could of course use an ordinary power series in E for the same purpose, but the maximum error for a Chebyshev-derived expansion of degree fifteen is a thousand times smaller than from the power series of $\sin(E)$ to the same order.

2. Chebyshev mechanics and a Matlab function

There are a couple of technical issues that are resolved by the following [10]:

Theorem 1.

(1) *For all* $\varepsilon \in [0, 1]$ *and for all* $M \in [-\pi, \pi]$, *there is a single real root which lies in the range* $E \in [-\pi, \pi]$.
(2) $E(\varepsilon, M + 2\pi k) = 2\pi k + E(\varepsilon, M)$; *equivalently, the function* $E(\varepsilon, M) - M$ *is periodic in* M *with period* 2π.

Because roots for $|M| > \pi$ can always be computed by solving Kepler's equation for $M \in [-\pi, \pi]$, we will henceforth restrict our efforts to this interval in M.

Second, a Chebyshev expansion in $T_j(x)$ is accurate only on the canonical interval $x \in [-1, 1]$. However, since the theorem shows that for $|M| < \pi$, the root lies on the interval $E \in [-\pi, \pi]$, we can stretch the Chebyshev interval to include all roots by expanding in $x \equiv E/\pi$.

Third, a polynomial of degree N has N roots: the last step in the algorithm is to discard all of these except for the single root that lies on the "good" interval for Chebyshev series, $x \in [-1, 1]$, or equivalently, to accept only the root on $E \in [-\pi, \pi]$.

A complete Matlab function to compute the Kepler root for $\varepsilon \leqslant 1$ and $M \in [-\pi, \pi]$ is given in Table 1. Its brevity—just seven lines—attests to the idea's simplicity. The approximation to Kepler's equation used in the code (degree 15) is

$$\begin{aligned} M = E - \varepsilon\Bigg\{&3.14\frac{E}{\pi} - 5.16\left(\frac{E}{\pi}\right)^3 + 2.55\left(\frac{E}{\pi}\right)^5 - 0.599\left(\frac{E}{\pi}\right)^7 \\ &+ 0.082\left(\frac{E}{\pi}\right)^9 - 0.0073\left(\frac{E}{\pi}\right)^{11} + 0.00046\left(\frac{E}{\pi}\right)^{13} - 0.000018\left(\frac{E}{\pi}\right)^{15}\Bigg\}. \end{aligned} \tag{4}$$

The maximum absolute error in computing the root of Kepler's equation by solving this proxy is only 4×10^{-10} for all $\varepsilon \in [0, 1]$ and for all $M \in [-\pi, \pi]$.

The mechanics of collecting powers of x to convert from Chebyshev series form to powers-of-E form is described in great detail in [6]; more general accounts of rootfinding-through-Chebyshev-expansions are given in [4,7,8].

Table 1
Matlab function to solve the Kepler equation using a polynomial of degree 15

```
function E = KeplerE(eccentricity, M);
Kepler = eccentricity*[0.00001877013878 0 −0.00046097562573 0
... 0.00736564609504 0 −0.08214347708860 0 0.59926386322604 0
... −2.55016394839721 0 5.16771277519855 0 −3.14159265354687 0];
Kepler(16) = −M, Kepler(15) = Kepler(15) + pi;
allrts = roots(Kepler), realrts = real(allrts(find(abs(imag(allrts)) < 1.E−10)));
smallrts = realrts(find(realrts< = pi)); E = pi*smallrts(find(smallrts> = −pi));
```

Table 2
Approximations and errors for small degree

Degree	$N=3$	$N=5$	$N=7$	$N=9$
$j=1$	8/3	3.112	3.1405	3.14156847
$j=3$	−8/3	−4.781	−5.1414	−5.1667199
$j=5$	–	1.669	2.4387	2.54332858
$j=7$	–	–	−0.43780	−0.58217893
$j=9$	–	–	–	0.064001762
Max. error.	0.37	0.080	0.0086	2.1×10^{-4}

Each column lists the coefficients of $(E/\pi)^j$ for the polynomial of degree N; the maximum error for the Kepler root by using the polynomial in place of $\sin(E)$ for that degree $\forall\varepsilon \leqslant 1$ and $\forall M \in [-\pi, \pi]$ is given in the bottom row.

Table 3
Approximations and errors for higher degree

Degree	$N=11$	$N=13$	$N=15$
$j=1$	3.14159226290564	3.14159264892171	3.14159265354687
$j=3$	−5.16768892929696	−5.16771238308857	−5.16771277519855
$j=5$	2.54992065480454	2.55015840469097	2.55016394839721
$j=7$	−0.59833380494771	−0.59923399525986	−0.59926386322604
$j=9$	0.08050047080247	0.08206587679402	0.08214347708860
$j=11$	−0.00599065426797	−0.00726109635030	−0.00736564609504
$j=13$	–	0.00039054429204	0.00046097562573
$j=15$	–	–	−0.00001877013878
Max. error.	3.3×10^{-6}	3.9×10^{-8}	4.2×10^{-10}

Same as previous table except for larger N.

3. Double Chebyshev series

Barker [1] expanded E as a double Chebyshev series in both the eccentricity ε and the mean anomaly M. (He actually expanded the "true anomaly" ν, which is known as an explicit analytic function of E, rather the eccentric anomaly E, but this distinction is irrelevant to the Chebyshev mechanics.) Unfortunately, for the largest eccentricity he considered, $\varepsilon = 0.975$, the errors were on the order of one part in a thousand even though his expansions included 361 terms in M and 79 in ε for a total of 28 519 terms in the two-dimensional series!

A fundamental difficulty with this sort of "direct" Chebyshev approximation of $E(\varepsilon, M)$, as opposed to the "polynomialization" of $\sin(E)$ described in the previous section, is that E is a function of two unknowns whereas the "polynomialization" method requires only a univariate Chebyshev series. A second reason for the length of Barker's approximations is that E is a periodic function of M; a Chebyshev series is inferior to Fourier series for periodic functions [5].

A third difficulty is that $E(\varepsilon, M)$ has a *triple root* at $\varepsilon = 1$, $M = 0$. This slows the convergence of the Chebyshev series in ε to a crawl when ε is close to one, at least for small M.

The polynomialization-of-the-sine method is free of this defect because the coefficients of the polynomial are smooth functions of ε and M even though the roots themselves are not. The zeros of the polynomial approximation to the Kepler equation move smoothly from simple root to near-triple zero to triple root without difficulty.

4. Alternative ways to "polynomialize" Kepler's equation

4.1. Power series of the sine

One may of course expand the sine function in Kepler's equation as a power series instead of a Chebyshev series and still "polynomialize" the equation. For example, with a power series truncated at degree fifteen, the maximum error for all $\varepsilon \in [0, 1]$ and all $M \in [-\pi, \pi]$ is only 3.3×10^{-7}. Unfortunately, the error from a power series is highly non-uniform, being as small as 10^{-16} for small M (where E is small) but rising to 10^{-7} or larger for all ε when M is close to π.

In contrast, the Chebyshev expansion is highly uniform [5] over the entire target interval in E, and this gives errors for the root that, while subject to small-scale fluctuations due to roundoff and so on, are similarly uniform over the entire target domain in ε and M. Furthermore, the maximum error for a Chebyshev-derived expansion of degree fifteen is a thousand times smaller than from the power series of $\sin(E)$ to the same order.

Because of the higher accuracy and greater uniformity of the Chebyshev series relative to the power series, a common strategy for improving approximations is "Lanczos economization of a power series" [16]: the replacement of a power series by a Chebyshev series of lower degree that retains the same accuracy.

4.2. Minimax approximations

A "minimax" polynomial approximation is defined to be that polynomial of a given degree N which, of all approximations to a given function $f(x)$ of that degree, gives the smallest value of the largest pointwise error on the specified interval in x. Such L_∞-norm-minimizing approximations can be calculated by the so-called Remez algorithm, which is available in many software libraries. However, it is well known [16,15] that truncated Chebyshev series are close to minimax approximations as expressed by the following well-known theorem, [15, p. 167]:

Theorem 2 *(Minimax/Chebyshev errors). Let $f_N(x)$ denote the Chebyshev series of $f(x)$, truncated after the Nth term, and let $f_N^{\text{minimax}}(x)$ denote the polynomial of smallest error in the L_∞ norm to the same function $f(x)$ among all the polynomials of degree N. Denote the errors by*

$$M_N \equiv \max_{x\in[-1,1]} \left|f(x) - f_N(x)\right|, \qquad C_N \equiv \max_{x\in[-1,1]} \left|f(x) - f_N^{\text{minimax}}(x)\right| \tag{5}$$

then

$$M_N \leqslant C_N \leqslant 4\left(1 + \frac{1}{\pi^2}\log(N)\right)M_N. \tag{6}$$

Since the error in the Chebyshev series of an analytic function falls *exponentially* with N whereas the improvement of the minimax approximation over the Chebyshev series grows only *logarithmically* with N, it follows that any minimax approximation can be matched in accuracy by a truncated Chebyshev series of slightly higher degree.

Because $\sin(\pi x)$ is an *entire* function, the difference between the minimax and Chebyshev approximations is especially small. First, the known large order asymptotics of the Bessel functions show the following.

Lemma 1. *Let $b_{2n+1}(z) \equiv 2J_{2n+1}(z)$ denote the absolute values of the Chebyshev coefficients of* $\sin(zx)$ *on $x \in [-1, 1]$. Then the ratio of successive non-zero coefficients is, in the asymptotic limit $n \to \infty$,*

$$\rho_n \equiv \frac{b_{2n+1}}{b_{2n+3}} \sim \frac{16}{z^2}n^2 + \mathrm{O}(n), \quad n \gg 1. \tag{7}$$

Recalling the magnitude of the nth term in a Chebyshev series is bounded by the magnitude of the coefficient, it follows that for $\sin(\pi x)$

$$C_{2N+1} \sim 2J_{2N+3}(\pi)\left\{1 + \mathrm{O}\left(\frac{1.6211}{N^2}\right)\right\} \approx 0.798\sqrt{\frac{1}{2N+3}}\left(\frac{4.27}{2N+3}\right)^{2N+3}. \tag{8}$$

This implies that to within $\mathrm{O}(1/N^2)$, the degree N Chebyshev approximation sees $\sin(\pi x)$ as a polynomial of degree $N+2$, and the error is dominated by $b_{N+2}T_{N+2}(x)$. However, the following theorem, a trivial generalization of one from [15, p. 170], asserts that minimax approximation cannot improve upon the Chebyshev series when approximating a polynomial with just one non-zero term of more than Nth degree:

Theorem 3. *Let $g(x) = P_N(x) + dT_k(x)$ where $P_N(x)$ is a polynomial of degree N and $k > N$. Then the Chebyshev series approximation of degree N to $g(x)$ and the minimax approximation of degree N to $g(x)$ are identical, and the error of both in the L_∞ norm is $|d|$.*

This leads to the conjecture that:

Conjecture 1. *The minimax error of degree N in approximating* $\sin(zx)$ *on $x \in [-1, 1]$ for any finite z is asymptotically smaller than the error in truncating the Chebyshev series after N terms by a fraction which is* $\mathrm{O}(1/N^2)$*, that is,*

$$\frac{C_N - M_N}{M_N} \sim \frac{\text{constant}}{N^2}, \quad N \to \infty. \tag{9}$$

Although we shall not provide a formal proof, numerical computations show that the relative minimax improvement, $(C_N - M_N)/M_N$, is only 3.6% for $N = 5$, 2.3% for $N = 7$, 0.73% for $N = 15$, the highest Chebyshev approximation considered here, and 0.34% for $N = 23$.

4.3. Rational approximations

$\sin(E)$ may also be approximated by a rational function. Both Pade–Chebyshev [12] and minimax rational approximants can be computed by well-known algorithms. Empirically, the "diagonal" approximants, that is, those for which the degrees of the numerator and denominator are chosen to be roughly equal, are usually the best and slightly superior to polynomial approximations with the same number of degrees of freedom.

For the Kepler equation, a rational approximation is appealing because, after multiplying through by the denominator, the Kepler equation is transformed into a *polynomial* equation (again!) whose degree is $\max(N_{\text{num}}, N_{\text{deg}} + 1)$ where N_{num} and N_{deg} are the degrees of the numerator and denominator of the rational approximation to $\sin(E)$. This is typically about half the degree of a polynomial-based approximation of similar accuracy.

Unfortunately, when the numerator is cleared, each coefficient of the resulting polynomial is a linear polynomial in ε. Thus, the rational-based approximation has roughly as many numerical coefficients as a Chebyshev polynomial-based scheme of the same accuracy since the latter requires only one numerical coefficient per power of E. Therefore, an exhaustive analysis of rational approximations does not seem warranted.

5. The Chebyshev–Frobenius matrix method

It is not strictly necessary to convert the "Chebyshevization" of Kepler's equation into an ordinary polynomial. Stetter [17] and Day and Romero [11,7] have shown that a polynomial equation in the form of a truncated Chebyshev series can be solved directly: the N roots are the *eigenvalues* of the Chebyshev–Frobenius matrix. If the expansion is truncated so that the equation is of degree N with Chebyshev coefficients a_j, then the matrix elements A_{jk} of the $N \times N$ matrix are, with δ_{jk} the usual Kronecker delta-function such that $\delta_{jj} = 1$ while $\delta_{jk} = 0$ if $j \neq k$,

$$A_{jk} = \begin{cases} \delta_{2,k}, & j = 1,\ k = 1, 2, \ldots, N, \\ \frac{1}{2}\{\delta_{j,k+1} + \delta_{j,k-1}\}, & j = 1, 2, \ldots, (N-1),\ k = 1, 2, \ldots, N, \\ (-1)\frac{a_{j-1}}{a_N} + (1/2)\delta_{k,N-1}, & j = N,\ k = 1, 2, \ldots, N. \end{cases} \tag{10}$$

The approximation to the Kepler equation is

$$\sum_{k=0}^{N} a_k T_k(E/\pi) = 0 \tag{11}$$

where the Chebyshev coefficients are

$$a_0 = M, \qquad a_1 = 2\varepsilon J_1(\pi) - \pi, \qquad a_{2k+1} = 2\varepsilon(-1)^k J_{2k+1}(\pi), \quad k = 1, 2, \ldots. \tag{12}$$

For a polynomialization of Kepler's equation of degree five, these general formulas yield

$$\left|\begin{array}{ccccc} 0 & 1 & 0 & 0 & 0 \\ (1/2) & 0 & (1/2) & 0 & 0 \\ 0 & (1/2) & 0 & (1/2) & 0 \\ 0 & 0 & (1/2) & 0 & (1/2) \\ (-1)\frac{M}{2\varepsilon J_5(\pi)} & (-1)\frac{\varepsilon J_1(\pi)-\pi/2}{\varepsilon J_5(\pi)} & 0 & \frac{\varepsilon J_3(\pi)}{\varepsilon J_5(\pi)} + (1/2) & 0 \end{array}\right|. \tag{13}$$

Only the root of the matrix that is on the interval $E \in [-\pi, \pi]$ is an approximation to the real-valued root of Kepler's equation for $M \in [-\pi, \pi]$.

This approximation of the roots of a transcendental equation by the eigenvalues of a matrix with matrix elements that are Bessel functions is quite amazing.

6. Analysis and conclusions

On the basis of formal operation count, the Chebyshev method is much slower than Newton's iteration. (The QR rootfinding method costs about $10N^3$ operations where N is the degree of the polynomial.) However, library polynomial rootfinders are highly optimized, and the differences between the Chebyshev and Newton Kepler-solvers will be smaller in practice.

Cost estimates for single-unknown-rootfinders have in any event been rendered obsolete by the speed of modern computers. Using either Newton's iteration with Charles and Tatum's initialization, or the Chebyshev strategy described here, one can solve the Kepler equation for thousands of different (ε, M) in under a second on a laptop computer.

Still, it is interesting that it is still possible to find new, robust algorithms even for this very old problem.

It is also interesting that it is not only easy to convert Kepler's equation into an equivalent polynomial equation with very high accuracy, but there are multiple ways to do so. The Chebyshev polynomial method of "polynomialization" of a transcendental equation is preferred here not because it is necessarily the most accurate, but because it is simple, highly uniform in E, and allows a detailed theoretical analysis.

A more important point is that transcendental equations arise in many forms in celestial mechanics and elsewhere. The Chebyshev method does *not* require a user-specified initial approximation to the root or roots and in this sense is a *non-iterative* strategy for rootfinding.[1] Chebyshev polynomialization is therefore even more attractive for other problems where an always-convergent initialization for Newton's iteration is as yet unknown.

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[1] Although most polynomial rootfinders do not require an approximation to the roots from the user, some may object to our adjective "non-iterative" since, hidden from the user, the QR method and the Jenkins–Traub scheme both iterate. However, few would insist on labelling the usual formula for the roots of the quadratic equation as "iterative" even though the square roots are always numerically evaluated by Newton iteration.

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