1. Consider the linear system \( \mathbf{Ax} = \mathbf{b} \) with

\[
\mathbf{A} = \begin{pmatrix}
77 & 7 & 10 & -13 & -3 & -7 & -17 & -4 & 6 & 9 \\
-9 & -74 & 1 & 8 & -20 & -8 & 6 & 6 & -4 & -3 \\
5 & 2 & -11 & -91 & 5 & -15 & 30 & 4 & 10 & -5 \\
3 & 4 & -3 & 14 & -89 & 4 & 11 & 13 & -1 & -28 \\
1 & -1 & -17 & 21 & -15 & -94 & 2 & -18 & 12 & 0 \\
2 & 1 & 5 & 5 & -2 & -3 & -80 & -15 & 14 & -21 \\
-5 & 0 & 8 & -5 & 1 & -2 & 23 & -67 & 17 & -1 \\
-14 & -4 & 5 & -31 & -10 & -3 & -19 & -21 & -127 & 19 \\
9 & -9 & 9 & 8 & 5 & 11 & -10 & -5 & -1 & -74
\end{pmatrix}
\]

and

\[
\mathbf{b} = (30, -3, 65, -54, -10, 96, 185, 7, 170, 89)^{\top}.
\]

(a) Is the matrix \( \mathbf{A} \) strictly diagonally dominant? (You may use your code \texttt{diagdom.m} from Homework #4 to determine this.)

(b) Try to solve the above system with both the Jacobi and Gauss-Seidel methods for a tolerance \( TOL = 10^{-10} \) and a maximum number of iterations \( N = 1000 \), for an initial guess

\[
\mathbf{x}_0 = (1, 1, 1, 1, 1, 1, 1, 1, 1)^{\top}.
\]

Do the two methods converge and, if so, what is the number of iterations required? What is the solution for \( \mathbf{x} \)? (You may use the book codes \texttt{ALG071.m} and \texttt{ALG072.m}, so that the iterations are not printed in the Matlab command window. After creating the augmented matrix in Matlab by \( \mathbf{M} = [\mathbf{A}; \mathbf{b}] \), the code \texttt{prep07.m} from the course website can be used to create an input file “data.txt” which has the required format for the book codes.)

(c) Explain how your results in (a),(b) are consistent with convergence theorems.

(d) Find the explicit expressions for the iteration matrices \( \mathbf{T}_J \) and \( \mathbf{T}_G \). (You will find it useful to use Matlab functions such as \texttt{diag}, \texttt{tril}, \texttt{triu}, and \texttt{inv}.)

(e) Using the results in (d), find the spectral radii \( \rho(\mathbf{T}_J) \) and \( \rho(\mathbf{T}_G) \). This can be done in Matlab with the command

\[
>> \text{rho} = \max(\text{abs(eig(T)))}
\]

Use these results to estimate the required number of iterations for each method. Do the results agree well with the actual number of iterations required?

1
2. Repeat Problem #1 for the linear equation $Ax = b$ with

$$A = \begin{pmatrix}
12 & 23 & 11 & 10 & 3 & -5 & 0 & -4 & -13 & -8 \\
-16 & 149 & 8 & -1 & 9 & -22 & -2 & 1 & -3 & -1 \\
16 & 6 & 133 & 7 & -4 & 2 & 8 & 18 & 6 & 5 \\
-14 & 6 & 0 & -81 & 12 & 5 & 14 & -18 & 2 & 1 \\
0 & -10 & -11 & 5 & -111 & -15 & 1 & -8 & -1 & 0 \\
-6 & 12 & -1 & -6 & 5 & -7 & 10 & 7 & -8 & -8 \\
-21 & 8 & -8 & -12 & 2 & -20 & -37 & 3 & -2 & 4 \\
-2 & 6 & -9 & -2 & -4 & -6 & -24 & -91 & -19 & 1 \\
-4 & -4 & 0 & 2 & 0 & -8 & -6 & -3 & 52 & 21 \\
-11 & 8 & 6 & 1 & -4 & -12 & -2 & -7 & -9 & -104
\end{pmatrix}$$

and

$$b = (47, 276, 187, -1, -165, 6, -20, -98, 19, -220)^\top.$$

3. Repeat Problem #1 for the linear equation $Ax = b$ with

$$A = \begin{pmatrix}
-40 & 12 & 1 & -1 & 6 & -11 & 23 & -11 & -10 & 2 \\
-3 & -47 & 10 & -33 & -4 & 17 & 8 & 17 & -1 & 9 \\
7 & 15 & 40 & 5 & -19 & 16 & 11 & -7 & 1 & 2 \\
5 & 12 & 9 & 18 & 18 & -13 & 4 & 14 & 15 & -6 \\
-7 & 2 & 5 & -1 & 104 & 4 & 1 & 26 & -13 & 6 \\
-7 & 6 & 1 & 10 & -6 & -11 & -2 & 8 & -1 & -1 \\
-7 & 0 & -11 & -4 & -3 & -12 & 21 & 22 & 14 & -16 \\
-6 & -15 & -1 & -17 & -7 & 20 & 7 & 25 & -9 & 20 \\
2 & -1 & -5 & -19 & -13 & -2 & -4 & -1 & 72 & 1 \\
7 & -1 & 1 & 8 & -3 & 10 & -11 & -1 & -18 & -190
\end{pmatrix}$$

and

$$b = (45, 38, -32, 37, 135, 11, 43, 29, -38, -13)^\top.$$
4. Consider the linear system \( \mathbf{Ax} = \mathbf{b} \) with

\[
\mathbf{A} = \begin{pmatrix}
2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 7 & -2 & -3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 8 & -3 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 7 -4 \\
0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 4
\end{pmatrix}
\]

and \( \mathbf{b} = (0, 4, 1, -1, 0, -1, 2, 1, 2, -3)^\top \).

(a) Construct the directed graph for the matrix \( \mathbf{A} \). Use the graph to determine whether the matrix is irreducible. If so, is it irreducibly diagonally dominant?

(b) Find the iteration matrix \( \mathbf{T}_J \) for the Jacobi method. Are all of its elements non-negative? Numerically determine the spectral radius of \( \mathbf{T}_J \). Is \( \rho(\mathbf{T}_J) < 1 \)?

(c) For \( \omega \in [0, 2] \), numerically calculate the spectral radii \( \rho(\mathbf{T}(\omega)) \) for the iteration matrices \( \mathbf{T}(\omega) \) in the SOR method. Use 2000 evenly spaced values \( \omega_i \) between 0 and 2 and calculate the spectral radius as in Problems #1-3. Plot \( \rho(\mathbf{T}(\omega)) \) versus \( \omega \). Do you observe a range \( 0 < \omega < \omega_0 \) with \( \omega_0 \geq 1 \) where \( \rho(\mathbf{T}(\omega)) \) is strictly decreasing? Use your results to estimate the optimal value \( \omega_b \) in SOR for this matrix.

(d) Try to solve the given problem \( \mathbf{Ax} = \mathbf{b} \) with the iteration methods of Jacobi, Gauss-Seidel, and SOR for the optimal parameter \( \omega_b \). Set the tolerance \( TOL = 10^{-10} \), the maximum number of iterations \( N = 1000 \), and use the initial guess \( \mathbf{x}_0 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^\top \). Give the final result for \( \mathbf{x} \) and state the number of iterations required for each method.

5. Consider the linear system \( \mathbf{Ax} = \mathbf{b} \) with

\[
\mathbf{A} = \begin{pmatrix}
4 & -4 & 2 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\
-4 & 5 & -1 & 0 & -2 & 1 & -1 & -1 & -1 & 0 \\
2 & -1 & 3 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & -1 & 0 & 2 & -1 & 0 & 1 \\
2 & -2 & 0 & -1 & 3 & -1 & -1 & -1 & -1 & 0 \\
-2 & 1 & -2 & 0 & -1 & 6 & 1 & 3 & 3 & 0 \\
0 & -1 & 0 & 2 & -1 & 1 & 7 & -2 & 1 & 3 \\
0 & -1 & -1 & -1 & -1 & 3 & -2 & 9 & 3 & 4 \\
0 & -1 & -1 & 0 & 1 & 3 & 1 & 3 & 5 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 3 & -4 & -1 & 7
\end{pmatrix}
\]
and
\[ \mathbf{b} = (4, -1, 8, 7, -2, -13, 12, -22, -13, 16)^	op. \]

(a) The matrix \( \mathbf{A} \) is clearly symmetric. Is it positive-definite? State how you reach your conclusion.

(b) As in Problem #4, numerically calculate the spectral radii \( \rho(\mathbf{T}(\omega)) \) for the iteration matrices \( \mathbf{T}(\omega) \) in the SOR method, for \( \omega \in [0, 2] \). Plot \( \rho(\mathbf{T}(\omega)) \) versus \( \omega \). Are your results in agreement with the Ostrowski-Reich Theorem? Use them to estimate the optimal value \( \omega_b \) in SOR for this matrix.

(c) Try to solve the given problem \( \mathbf{A}\mathbf{x} = \mathbf{b} \) with the iteration methods of Jacobi, Gauss-Seidel, and SOR for the optimal parameter \( \omega_b \). Set the tolerance \( TOL = 10^{-10} \), the maximum number of iterations \( N = 5000 \), and use the initial guess \( \mathbf{x}_0 = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^	op \). Give the final result for \( \mathbf{x} \) and state the number of iterations required for each method that converges.

6. Give a direct proof (not using the Ostrowski-Reich theorem) that, if the matrix \( \mathbf{A} \) is self-adjoint, positive-definite, then Gauss-Seidel iteration converges for any initial data \( \mathbf{x}_0 \).

\textit{Hint:} Let \( \mathbf{A} = \mathbf{D} - \mathbf{L} - \mathbf{L}^* \), \( \mathbf{T}_G = (\mathbf{D} - \mathbf{L})^{-1}\mathbf{L}^* \). Consider the matrix \( \mathbf{T}_1 = \mathbf{D}^{1/2}\mathbf{T}_G\mathbf{D}^{-1/2} = (\mathbf{I} - \mathbf{L}_1)^{-1}\mathbf{L}_1^* \), where \( \mathbf{L}_1 = \mathbf{D}^{-1/2}\mathbf{L}\mathbf{D}^{-1/2} \). Prove that \( \rho(\mathbf{T}_1) < 1 \).

7. This problem considers the spatial discretization of the Poisson equation on the unit square
\[
\begin{aligned}
-\Delta u(x, y) &= f(x, y), \quad 0 < x, y < 1, \\
u(x, y) &= 0, \quad x = 0, 1 \text{ or } y = 0, 1,
\end{aligned}
\]
as discussed in the course notes, which transforms it into a linear system \( \mathbf{A}\mathbf{u} = \mathbf{f} \). This equation describes, for example, the steady-state distribution of temperature \( u(x, y) \) in a square domain, if there is a spatial heating source \( f(x, y) \) and if the walls of the domain have temperature held fixed at 0.

(a) Find an explicit formula for \((\mathbf{T}_J\mathbf{u})_{ij}\) in terms of the quantities \( u_{i'j'} \), \( 1 \leq i', j' \leq n \).

(b) Verify that the eigenfunctions of \( \mathbf{T}_J \) are of the form
\[
u_{i,j}^{(k,l)} = \sin \left( \frac{\pi k i}{n+1} \right) \sin \left( \frac{\pi l j}{n+1} \right)
\]
with the stated eigenvalues. Use your results to verify the expression for \( \rho(\mathbf{T}_J) \).

(c) As discussed in class, the matrix \( \mathbf{A} \) has property \( A \). It also satisfies \( \rho(\mathbf{T}_J) < 1 \). Why must all eigenvalues of \( \mathbf{T}_J \) be real? Because of these facts, the Young-Varga
theorem applies. Use the theoretical results for $\rho(T_J)$ and $\rho(T(\omega_b))$ to verify that

$$\ln \varrho(T_J) = -\frac{\pi^2}{2(n+1)^2} + O\left(\frac{1}{n^4}\right), \quad \ln \varrho(T(\omega_b)) = -\frac{2\pi}{n+1} + O\left(\frac{1}{n^3}\right).$$

Using this, explain why optimal SOR is approximately $n$ times faster than Jacobi.

(d) Create the $n^2 \times n^2$ block-tridiagonal matrix $A$ for $n = 37$ using the Matlab function $A=gallery(\text{`poisson'},n)$. Next create an $n^2$-dimensional column vector $f$ for $n = 35$ corresponding to a source function of the form

$$f(x, y) = \begin{cases} g(x)h(y) & x \leq y \\ 0 & x > y \end{cases}$$

with

$$g(x) = \begin{cases} 1 & |x - 0.5| < 0.25 \\ 0 & |x - 0.5| \geq 0.25 \end{cases}, \quad h(y) = \begin{cases} 1 & |y - 0.5| < 0.25 \\ 0 & |y - 0.5| \geq 0.25 \end{cases}.$$  

You will need to stack the columns of the matrix $F_{ij} = f(x_i, y_j), 1 \leq i, j \leq n$ to form the vector $f$. You can use the Matlab function reshape to do this.

Finally, use the book codes ALG072.m and ALG073.m to solve the linear system $Au = f$ using Gauss-Seidel and optimal SOR. Set tolerance $TOL = 10^{-8}$ and maximum number of iterations $N = 5000$. Take as your initial guess $u_0 = \|f\|_2A^Tf$, which would be the exact solution if $f$ were an eigenvector of $A$. (To store the augmented matrix $M$ into an input file using prep07.m, you will need to declare the matrix to be full, since $A=gallery(\text{`poisson'},n)$ creates a sparse matrix. Use the Matlab command full($M$). Note that this is actually a very stupid thing to do for a real-world problem, since the main advantage of Gauss-Seidel and SOR for these problems is that $A$ is sparse, which makes multiplications by $A$ cheap! However, the way the book codes are written requires full matrices.)

How many iterations are required for Gauss-Seidel and optimal SOR to converge?

Plot your approximate solution $u(x, y)$ as a function of $(x, y)$ in the unit square. To do this, you will need to use reshape to transform the $n^2$-dimensional column vector $u$ into an $n \times n$ matrix $U$. Your results will look better if you transform matrix $U$ to be $(n+2) \times (n+2)$ matrix by adding “border” values of 0 as additional rows top and bottom and columns left and right. You can then plot your results in Matlab with:

```matlab
>> x=0:1/n:1; y=x;
>> surf(x,y,U.')
```

(Note that, as a legacy of Matlab’s early relation to Fortran programming, one must take a transpose of the matrix $U$ to plot its entries with surf.)

In the same manner, transform your initial guess $u_0$ into an $n \times n$ matrix $U_0$ and plot it also with surf. How does the final approximate solution $u(x, y)$ of the Poisson equation differ from the initial guess $u_0(x, y)$? Do your results make physical sense?