Problem 1. (a) Per the hint, write $\frac{dx}{dv} = \dot{x} = -\frac{v}{\omega^2}x$ or $-\omega^2 x \, dx = v \, dv$. Integrate to get $-\omega^2 x^2 / 2 = v^2 / 2 + C'$ for some $C'$. Letting $C := -2C'$ we find that $C = \omega^2 x^2 + v^2 \geq 0$.

(b) Recalling that total energy $E$ is kinetic energy $E_K$ plus potential energy $E_P$, where $E_K = \frac{1}{2}mv^2$ and $E_P = \frac{1}{2}kx^2$, we have $E = \frac{1}{2}(mv^2 + kx^2)$. But recalling that by definition $k = m\omega^2$, we can rewrite $E = \frac{1}{2}(v^2 + \omega^2 x^2) = \frac{1}{2}C$, which is constant in time under the flow.

Problem 2. In matrix form, write

$$
\begin{bmatrix}
\dot{x} \\
\dot{y}
\end{bmatrix}
= 
\begin{bmatrix}
-1 & 1 \\
1 & -1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}.
$$

To find the vector field, evaluate the pair $(\dot{x}, \dot{y})$ for various $(x, y)$.

Problem 3. (a) Following the hint, we have $\dot{D} = \dot{x} - \dot{y} = 2(y - x) = -2D$. Evidently $D = 0$ or $y = x$ is a stable fixed point of $D$. In other words, for any starting position the trajectory tends towards the line $y = x$. Next, observe $\dot{S} = \dot{x} + \dot{y} = 0$. In other words, the value $x + y$ doesn’t vary. If the trajectories were anything other than a straight line, this couldn’t be the case. This suggests I could draw a circle around any point on the line $y = x$ and guarantee that the trajectories move in a straight path towards said line without leaving the circle. I sketch this state of affairs below. We expect then that the system is Lyapunov stable.
(b) Setting \( R := r^2 \) and following the hint, we have \( \dot{R} = 2x \dot{x} + 2y \dot{y} = -2(x^2 + y^2) = -2R \). As before we note this has a stable fixed point for \( R = 0 \), or in other words, when \( x^2 + y^2 = 0 \), which is only possible when \( (x, y) = 0 \). We expect that all trajectories in this vector field thus pool at the origin, which is asymptotically stable.

**Problem 4.** (a) Consider the line \( \ell := \{(x, y) : y = x\} \). From the dynamics of \( \dot{S} \) all trajectories move towards \( \ell \). From the dynamics of \( S \) these trajectories are orthogonal to \( \ell \), since they form segments of the lines \( \ell_\delta := \{(x, y) : y = S - x\} \) for \( S > 0 \). Now let \( \varepsilon > 0 \), take \( \delta := \varepsilon \), and pick a \( \delta \)-ball around some \( (x, y) \in \ell \). Choose any point \( (x_0, y_0) \) in the \( \delta \)-ball. The trajectory of a system starting at \( (x_0, y_0) \) is necessarily along \( \ell_\delta \) for \( S = x_0 + y_0 \), namely the segment from \( (x_0, y_0) \) to \((x^*, y^*)\) where \((x^*, y^*) \in \ell \cap \ell_\delta \) is the unique element of the singleton intersection. Consider the right triangle formed by \((x, y), (x^*, y^*), \) and \((x_0, y_0)\). Its hypotenuse, the edge from \((x, y)\) to \((x^*, y^*)\), and so \((x^*, y^*)\) is also contained in the \( \delta \)-ball. Since the \( \delta \)-ball is convex, the edge from \((x_0, y_0)\) to \((x^*, y^*)\) is contained within the \( \delta \)-ball and thus also the \( \varepsilon \)-ball around \((x, y)\), and we’re done.

(b) To prove Lyapunov stability it suffices to take \( \delta := \varepsilon > 0 \). Consider a system starting at \((x_0, y_0)\) in the \( \delta \)-ball around the origin. Since \( \forall t : R(t) \geq 0 \), we have \( \forall t : \dot{R}(t) \leq 0 \). In other words, \( \forall t : x(t)^2 + y(t)^2 \leq x_0^2 + y_0^2 < \delta \). This places the trajectory of \((x, y)\) in the \( \delta \)-ball and thus also the \( \varepsilon \)-ball of the origin. This gives Lyapunov stability. Next we prove that the origin is attracting. Recall that the origin is a stable fixed point for \( \dot{R} = -2R \), so we have \( R(t) \to 0 \) as \( t \to \infty \) for any initial \( R(0) \). In particular for any \((x_0, y_0)\) in any \( \delta \)-ball about the origin we have \( x(t)^2 + y(t)^2 \to 0 \), giving us as well that \((x(t), y(t)) \to 0 \). Since the origin is both Lyapunov stable and attracting, it is asymptotically stable.

**Problem 5.** (a) Write

\[
\mathbf{x} = A \mathbf{x} = \begin{bmatrix} 1 & -3 \\ 0 & -5 \end{bmatrix} \mathbf{x}.
\]

The characteristic polynomial of \( A \) is \( \lambda^2 + 4\lambda + 13 \), which has roots \( \lambda = -2 \pm 3i \). These are \( A \)'s eigenvalues with respective eigenvectors \((1, 1 \pm i)\).

(b) From basic ODE theory we can write a general complex form

\[
\mathbf{x}(t) = c_1 e^{(-2+3i)t} \begin{bmatrix} 1 \\ 1-i \end{bmatrix} + c_2 e^{(-2-3i)t} \begin{bmatrix} 1 \\ 1+i \end{bmatrix} = (c_1 + c_2)e^{-2t} \begin{bmatrix} \cos(3t) \\ \cos(3t) + \sin(3t) \end{bmatrix} + i(c_1 - c_2)e^{-2t} \begin{bmatrix} \sin(3t) \\ \sin(3t) - \cos(3t) \end{bmatrix}
\]

for some constants \( c_1, c_2 \), the last equality attained by applying Euler’s formula \( e^{i\theta} = \cos(\theta) + i\sin(\theta) \) and doing some algebra. We only care about the real part, namely

\[
\mathbf{x}(t) = \text{Re} \{ \mathbf{x}(t) \} = \text{Re}(c_1 + c_2)e^{-2t} \begin{bmatrix} \cos(3t) \\ \cos(3t) + \sin(3t) \end{bmatrix} - \text{Im}(c_1 - c_2)e^{-2t} \begin{bmatrix} \sin(3t) \\ \sin(3t) - \cos(3t) \end{bmatrix}.
\]

(c) As both eigenvalues are complex we expect a spiral. In particular since their real parts are negative, we expect a stable, decaying spiral. This is borne out in our plot.
(d) Returning to our equation in (b) we write

\[ x(0) = \text{Re}(c_1 + c_2) \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \text{Im}(c_1 - c_2) \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}. \]

This system can be solved to get \( \text{Re}(c_1 + c_2) = 3 \) and \( \text{Im}(c_1 - c_2) = 1 \), so that

\[ x(t) = 3e^{-2t} \begin{bmatrix} \cos(3t) \\ \cos(3t) - \sin(3t) \end{bmatrix} - e^{-2t} \begin{bmatrix} \sin(3t) \\ \sin(3t) + \cos(3t) \end{bmatrix}. \]

**Problem 6.** (a) Write

\[ \dot{x} = Ax = \begin{bmatrix} -11 & -7 \\ 14 & 10 \end{bmatrix} x. \]

The characteristic polynomial of \( A \) is \( z^2 + z - 12 \), which has roots \( z = 3, -4 \). These are \( A \)'s eigenvalues with respective eigenvectors \((-1, 2), (-1, 1)\).

(b) From basic ODE theory we can immediately write

\[ x(t) = c_1 e^{3t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 e^{-4t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \]

for some constants \( c_1, c_2 \).

(c) We have \( \det(A) < 0 \) and eigenvalues of different signs, and so we expect a saddle. Again this is borne out in our graph.
Returning to our form in (b) we write
\[ x(0) = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}. \]
This system can be solved to get \( c_1 = 7 \) and \( c_2 = 10. \)

**Problem 7.** (a) Let \( x := I \) and rewrite this ODE as the system
\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -\frac{x}{CL} - \frac{Ry}{L},
\end{align*}
\]
or alternatively,
\[
\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/CL & -R/L \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.
\]

(b) First, \( \det(A) = 1/CL > 0, \) excluding saddles and non-isolated fixed points. Then \( \text{tr}(A) = -R/L. \) When \( R > 0 \) we have \( \text{tr}(A) < 0, \) giving us either stable spirals or stable nodes, both of which are asymptotically stable. When \( R = 0 \) we have \( \text{tr}(A) = 0, \) giving us centers, which are neutrally stable.

(c) **When \( R^2C - 4L > 0 \).** Noting that the characteristic function of \( A \) is \( z^2 + (R/L)z + 1/CL \) which has roots \((-R/L \pm \sqrt{R^2/L^2 - 4/CL})/2, \) it suffices for us to look at the term \( R^2/L^2 - 4/CL = (R^2C - 4L)/CL^2 \) to classify the fixed point. If by assumption it’s positive then we have two real eigenvalues for \( A, \) both with the same sign. They cannot be positive since their sum is \( \text{tr}(A), \) which would require that \( R < 0, \) nor can one be 0 or else their product \( \det(A) \) vanishes, a contradiction. Thus they must be negative and our fixed point is a stable node.

When \( R^2C - 4L = 0. \) Then we have just one root, one eigenvalue, and thus one eigenvector, giving us a degenerate node.

When \( R^2C - 4L < 0. \) Then we have two complex roots. We’ve already determined that if \( R = 0 \) we attain a center and thus a neutrally stable system. If \( R > 0 \) then \( \text{tr}(A) < 0 \) and our eigenvalues have strictly existent, negative real parts. Thus we have a stable spiral.
(a) \( R^2 C - 4L > 0 \)

(b) \( R^2 C - 4L = 0 \)

(c) \( R^2 C - 4L < 0 \)