Midterm 550.391, Oct. 12, 2009.

Do all of the following **three** *problems.* Show all your work. Answers without supporting work may receive no credit.

I attest that I have completed this exam without unauthorized assistance from any person, materials, or device:

Full name: KEY

Signature: ______________________

(See the Johns Hopkins Handbook *Academic Ethics for Undergraduates*).

1. Consider the vector field

$$\dot{x} = 1 - 2x + x^2$$

on the whole real line $-\infty < x < +\infty$.

(a) Find all the fixed points of the system.

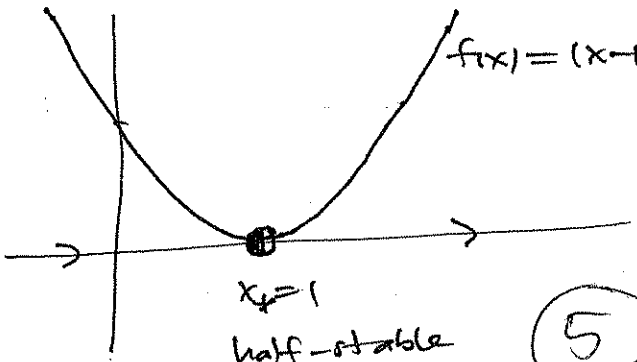
(b) Determine the stability of each fixed point. State whether linearization is useful or not for this purpose.

(c) Calculate the potential $V(x)$ for the system and sketch its graph.

(d) Sketch the phase portrait of the system on the real line.

(a) $0 = 1 - 2x + x^2 = (x-1)^2 \Rightarrow x_* = 1$ 8

(b)



5

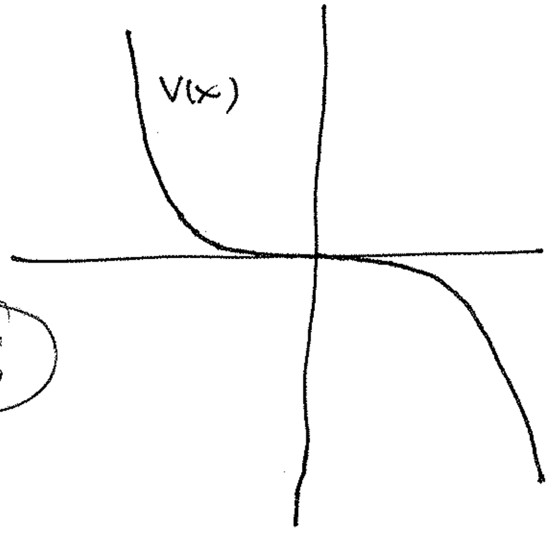
linearization: $x = 1 + \delta x$

$\dot{\delta x} = \dot{x} = (x-1)^2 = (\delta x)^2 \cong 0$

$\therefore$ linearization gives no information about stability 4

(c) $V(x) = -\int f(x)\,dx$

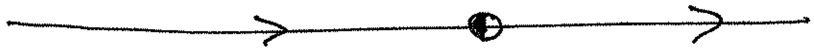
$= -\frac{1}{3}(x-1)^3$



$x=1$

8

(d)



$x_* = 1$

8

2. Consider the equation

$$\dot{x} = rx - \cosh x$$

with parameter r, on the interval $-\infty < x < +\infty$.

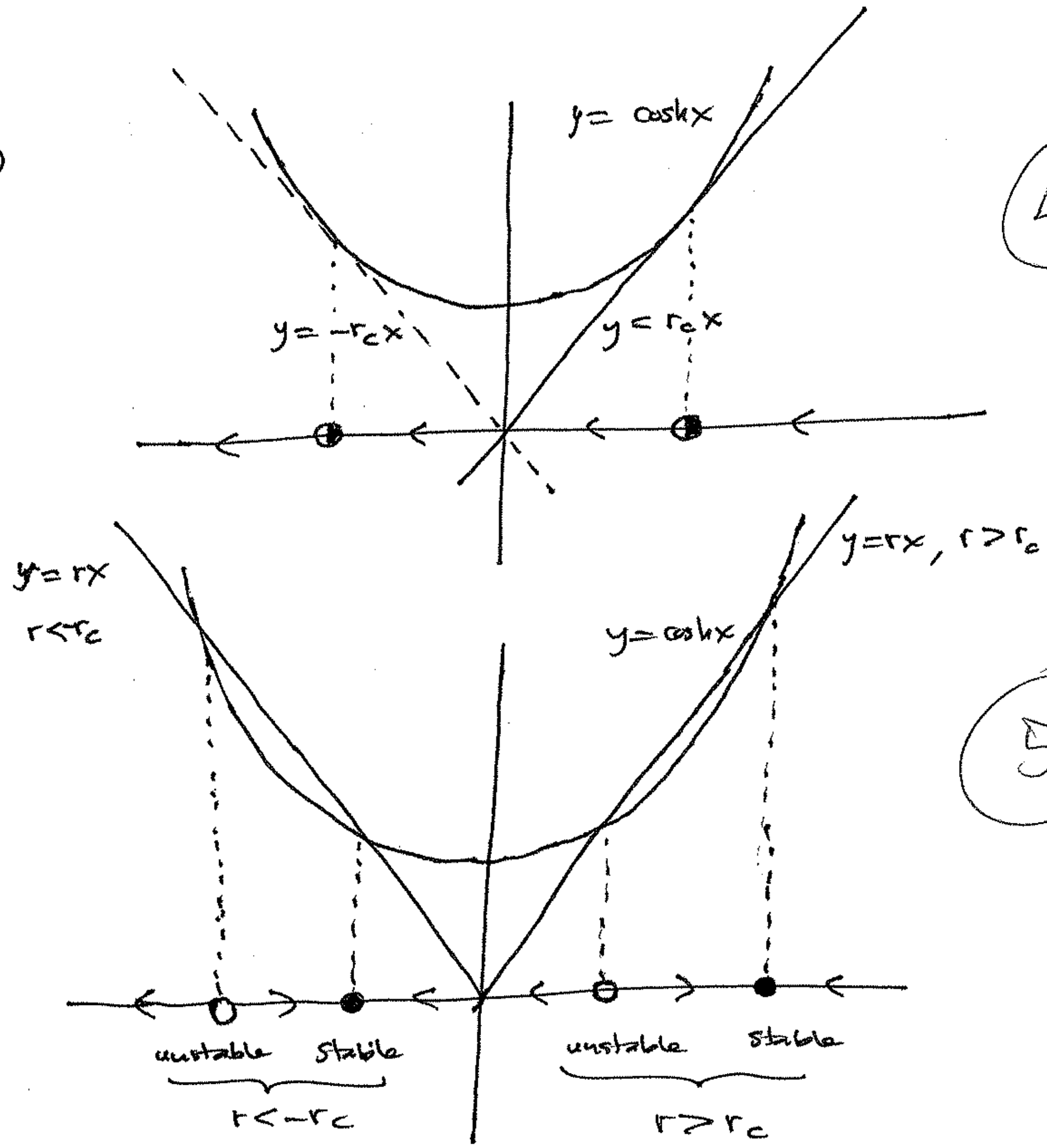
(a) Show graphically that bifurcations occur in this system at two critical values (r_c, x_c). Determine the stability of any fixed points that occur for all values of r.

(b) Derive an analytical equation for the values x_c (but don't solve the equation).

(c) Identify the type of bifurcations and qualitatively sketch the bifurcation diagram in the (r, x) plane.

(d) Derive a formula $r(x)$ that quantitatively gives the bifurcation diagram.

(a)



4

5

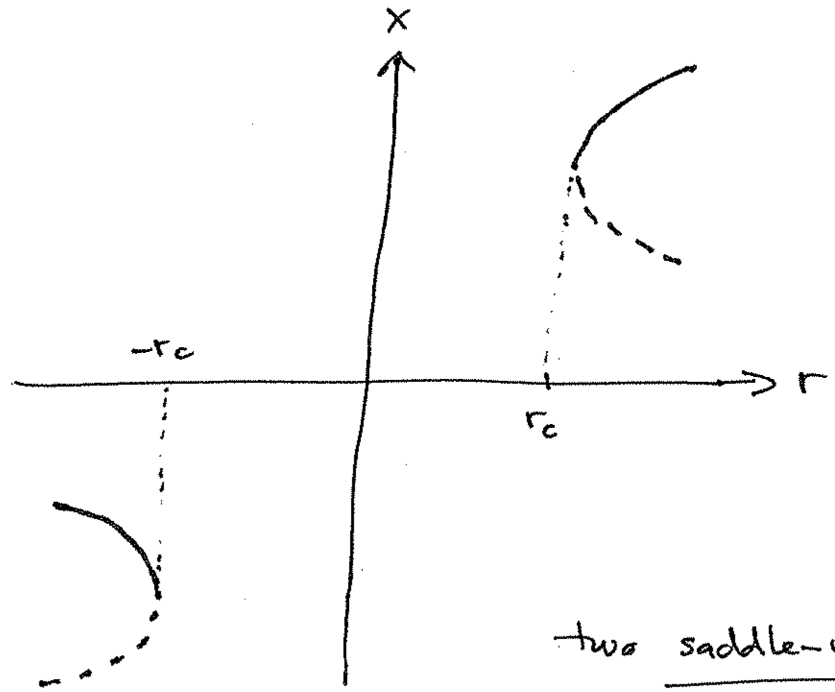
(b) $rx - \cosh x = 0 \implies rx = \cosh x$

$\frac{\partial}{\partial x}(rx - \cosh x) = 0 \implies r = \sinh x$

8

$\therefore\ r_c \sinh x_c = \cosh x_c$ or $\underline{x_c \tanh x_c = 1}$

(c)



8

two $\underline{\text{saddle-node bifurcations}}$

(d) $rx - \cosh x = 0$

$$\implies r(x) = \frac{\cosh x}{x}$$

8

3. Consider the system

$$\dot{x} = (\cos t)x.$$

(a) Verify that the periodic function $x(t) = x_0 \exp(\sin t)$ is the exact solution for initial condition x_0.

(b) Show that the system has a time-dependent potential $V(x,t)$ such that

$$\dot{x} = -\frac{\partial V}{\partial x}(x,t)$$

and find the explicit formula for $V(x,t)$.

(c) Calculate the time-derivative $(d/dt)V(x(t),t)$. Use this result to explain why there is no contradiction between (a) and (b).

(a) $\frac{d}{dt} x = x_0 \frac{d}{dt} e^{\sin t} = x_0 \cdot e^{\sin t} \cdot \cos t = (\cos t) \cdot x$ ✓

$x(t) = e^{\sin t}$ is 2π-periodic

10

(b) $V(x,t) = -\int dx\, f(x,t) = -\frac{1}{2}(\cos t)\, x^2$

$\Longrightarrow \quad \dot{x} = -\frac{\partial V}{\partial x} = (\cos t) \cdot x$ ✓

10

(c) $\frac{d}{dt} V(x,t) = \frac{\partial V}{\partial x} \cdot \dot{x} + \frac{\partial V}{\partial t}$

$= -(\cos t)x \cdot (\cos t)x + \frac{1}{2}(\sin t)\, x^2$

$= \left(\frac{1}{2} \sin t - \cos^2 t\right) x^2$

6

(cont'd)

Note that $\frac{d}{dt} V(x,t)$ can be <u>positive</u>. For example, if $t = \pi/2$

$$\frac{1}{2}\sin t - \cos^2 t = \frac{1}{2} - 0 = \frac{1}{2} > 0$$

The proof that a system with a potential has no periodic solutions is based on the fact that $\frac{d}{dt} V(x(t)) \leq 0$. Since this need not be true for a time-dependent potential, there is no contradiction!

7