Midterm 550.391, Oct. 4, 2010.

Do all of the following **three** *problems.* Show all your work. Answers without supporting work may receive no credit.

I attest that I have completed this exam without unauthorized assistance from any person, materials, or device:

Full name: KEY

Signature: ______________________________

(See the Johns Hopkins Handbook *Academic Ethics for Undergraduates*).

1. Consider the vector field

$$\dot{x} = 1 + \cos x$$

on the whole real line $-\infty < x < +\infty$.

(a) Find all of the fixed points of the system.

(b) Calculate the linearized dynamics around each fixed point. What does linearization imply for their stability?

(c) Using the vector field $f(x) = 1 + \cos x$, sketch the phase portrait of the system. What does the portrait imply for the stability of the fixed points?

(a) $1 + \cos x = 0 \implies \cos x = -1$

$\implies x = \pi + 2\pi k \,,\ k = 0, \pm 1, \pm 2, \ldots$

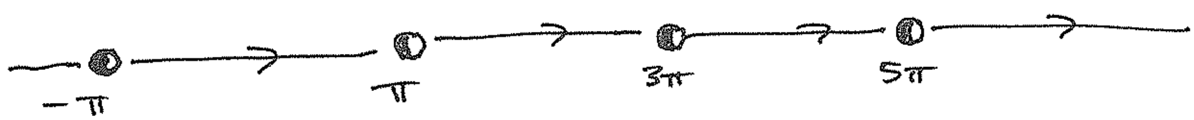
$\implies x_k = (2k+1)\pi \,,\ k = 0, \pm 1, \pm 2, \ldots$

(b) $f(x) = 1 + \cos x \implies f'(x) = -\sin x$

$f'(x_k) = -\sin \pi = 0$

$\therefore\ \delta \dot{x} = 0$; linearization implies nothing about stability !

(c) $f(x) = 1 + \cos x \geq 0$ since $\cos x \geq -1$. Thus, all points move to the right!



All equilibrium points are semi-stable

2. Consider the equation

$$\dot{x} = x^{1/5}$$

on the interval $-\infty < x < +\infty$ with a single equilibrium point at $x_* = 0$. Note that $x^{1/5}$ is the real 5th root, which satisfies $(-x)^{1/5} = -x^{1/5}$.

(a) Does the linearization exist at $x_* = 0$? If so, calculate it. Does the potential $V(x)$ exist? If so, calculate and sketch it.

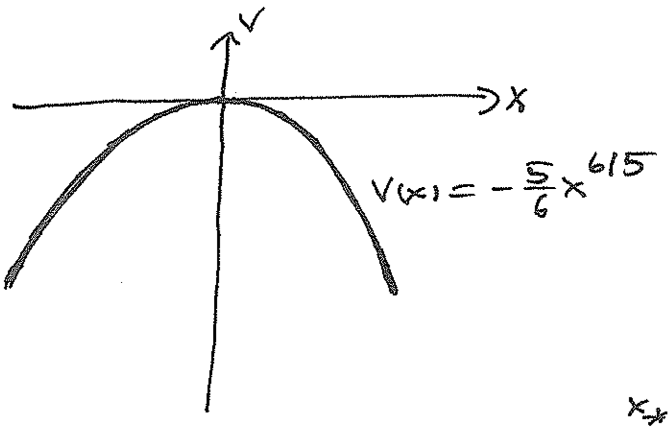
(b) Use the results in (a) to determine the stability of $x_* = 0$ and to sketch the phase portrait of the system.

(c) Calculate the time $T(\epsilon)$ for a solution to start at $x = \epsilon > 0$ and end at $x = 1$ and then find $T(0) = \lim_{\epsilon \to 0} T(\epsilon)$. But the equilibrium point $x_* = 0$ stays fixed for all times, so shouldn't $T(0) = +\infty$? Explain this apparent contradiction.

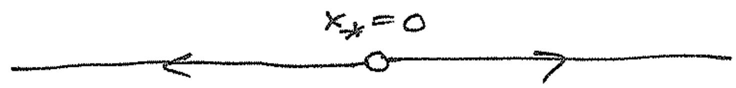
(a) $f(x) = x^{1/5} \Rightarrow f'(x) = \frac{1}{5} x^{-4/5} \Rightarrow f'(0) = +\infty$

$\therefore$ linearization at $x_* = 0$ does not exist

$$V(x) = -\int f(x)\,dx = -\frac{5}{6} x^{6/5} + C$$



(b) The origin is unstable :



(c) $$T(\epsilon) = \int_0^{T(\epsilon)} dt = \int_\epsilon^1 \frac{dx}{\dot{x}} = \int_\epsilon^1 \frac{dx}{x^{1/5}} = \frac{5}{4} x^{4/5} \Big|_\epsilon^1 = \frac{5}{4}\left(1 - \epsilon^{4/5}\right)$$

$T(0) = \lim_{\epsilon \to 0} T(\epsilon) = \frac{5}{4}$. There is no contradiction because the solution is non-unique for initial condition $x(0) = 0$. For any time τ, $0 \le \tau \le +\infty$, there is a solution which stays at $x = 0$ for a time τ before moving away.

3. Consider the equation

$$\dot{x} = r^2 - x^2$$

with parameter r, on the interval $-\infty < x < +\infty$.

(a) Find all of the fixed points of this system for each real value of r and find the critical value $r = r_c$ at which a bifurcation occurs.

(b) Sketch the phase portraits for $r < r_c$, $r = r_c$, and $r > r_c$ and determine the stability of all of the fixed points in each case.

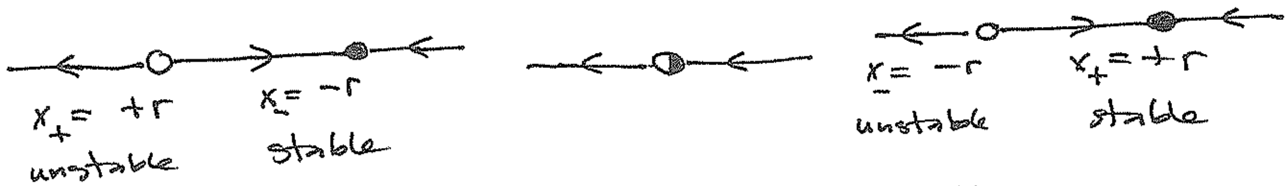
(c) Identify the type of bifurcation and sketch the bifurcation diagram.

(a) $\dot{x} = r^2 - x^2 = (r+x)(r-x) = 0$

$\Longrightarrow$ $\underline{x_\pm(r) = \pm r}$

Bifurcation occurs at $\underline{r_c = 0}$ when $x_+ = x_- = 0$

(b) $r < r_c = 0$ $\qquad$ $r = r_c = 0$ $\qquad$ $r > r_c = 0$



(c) $\underline{\text{transcritical bifurcation}}$

bifurcation diagram