Midterm 550.391, Oct. 3, 2011.

Do all of the following **three** *problems.* Show all your work. Answers without supporting work may receive no credit.

I attest that I have completed this exam without unauthorized assistance from any person, materials, or device:

Full name: SOLUTION KEY

Signature: G. Eyink

(See the Johns Hopkins Handbook *Academic Ethics for Undergraduates*).

1. Consider the vector field

$$\dot{x} = x^{1/2} - x$$

on the non-negative real line $0 \leq x < +\infty$.

(a) Find all of the fixed points of the system.

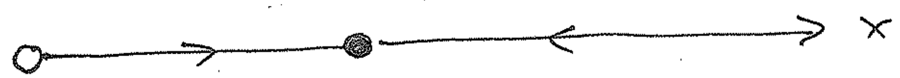
(b) Sketch the phase portrait of the system and use your sketch to classify the stability of the fixed points.

(c) At each fixed point, either calculate the linearization of the equation or else explain why this is not possible. Explain how your results are consistent with those in (b).

(a) $x^{1/2} - x = 0 \implies x^{1/2} = x \implies x = 0$ or $x = 1$

(b) $\dot{x} \cong x^{1/2}$ for $x \ll 1$

$\dot{x} \cong -x$ for $x \gg 1$



$x_* = 0$ <u>unstable</u> $\qquad x_* = 1$ <u>stable</u>

(c) $f'(x) = \frac{1}{2}x^{-1/2} - 1 \implies f'(0) = \infty$. linearization does not exist!
(no information on stability)

$f'(1) = \frac{1}{2} - 1 = -\frac{1}{2} < 0$

$\implies x_* = 1$ is <u>stable</u>, consistent with (b)

2. Consider the dynamical system defined by the potential

$$V(x) = \frac{1}{4}(x^2 - r)^2$$

on the interval $-\infty < x < +\infty$.

(a) Find the extrema of the potential and identify them as maxima or minima, for each of the cases $r < 0$, $r = 0$, and $r > 0$. Using this information, sketch the potential in each of these three cases.

(b) Use part (a) to sketch a bifurcation diagram for this system. What type of bifurcation do you find?

(a) $V'(x) = \frac{1}{4} \cdot 2(x^2 - r) \cdot 2x = x(x^2 - r) = 0$

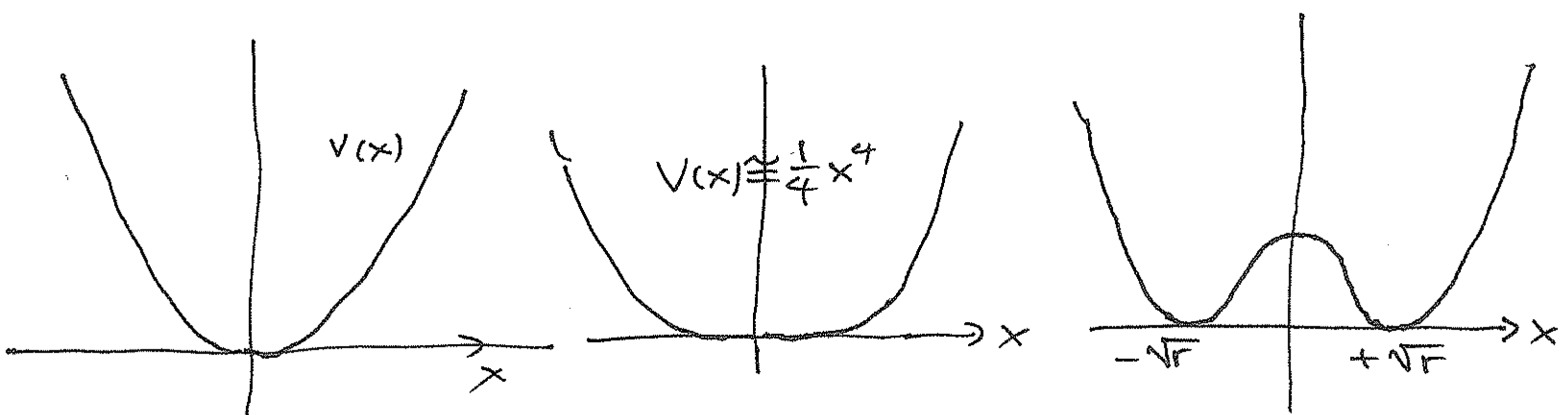
$\Longrightarrow$ extrema $\underline{x = 0}$, $\underline{x = \pm\sqrt{r}}$ (for $r > 0$)

$V''(x) = 3x^2 - r$

$\therefore V''(0) = -r \Longrightarrow x = 0$ minimum for $r < 0$
maximum for $r > 0$

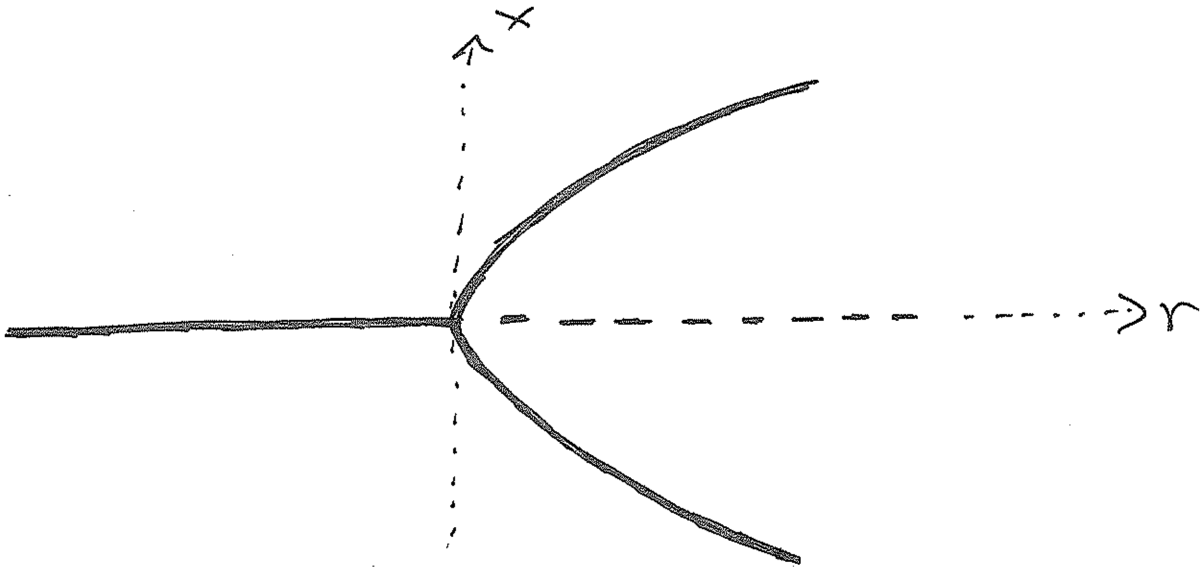
$V''(\pm\sqrt{r}) = 3r - r = 2r \Longrightarrow x = \pm\sqrt{r}$ minima for $r > 0$

$r < 0$ $\qquad$ $r = 0$ $\qquad$ $r > 0$



(b) Since minimum of $V \Rightarrow$ stable

maximum of $V \Rightarrow$ unstable



BIFURCATION DIAGRAM

This is a supercritical pitchfork bifurcation

3. Consider the equation

$$\dot{x} = f(x, r) = e^x - rx$$

with parameter $r > 0$, on the interval $-\infty < x < +\infty$.

(a) What two conditions are required for $f(x, r)$ to have a bifurcation at a point (x_c, r_c)? Use these conditions to locate the bifurcation for the given system.

(b) Plot e^x and rx for different values of $r > 0$ with $r < r_c$, $r = r_c$, and $r > r_c$. Use these plots to identify the type of bifurcation and sketch the bifurcation diagram.

(a) root condition : $f(x_c, r_c) = 0$

tangency condition : $\frac{\partial f}{\partial x}(x_c, r_c) = 0$

$f(x,r) = e^x - rx = 0$

$\Longrightarrow rx = r$

$\frac{\partial f}{\partial x}(x,r) = e^x - r = 0$

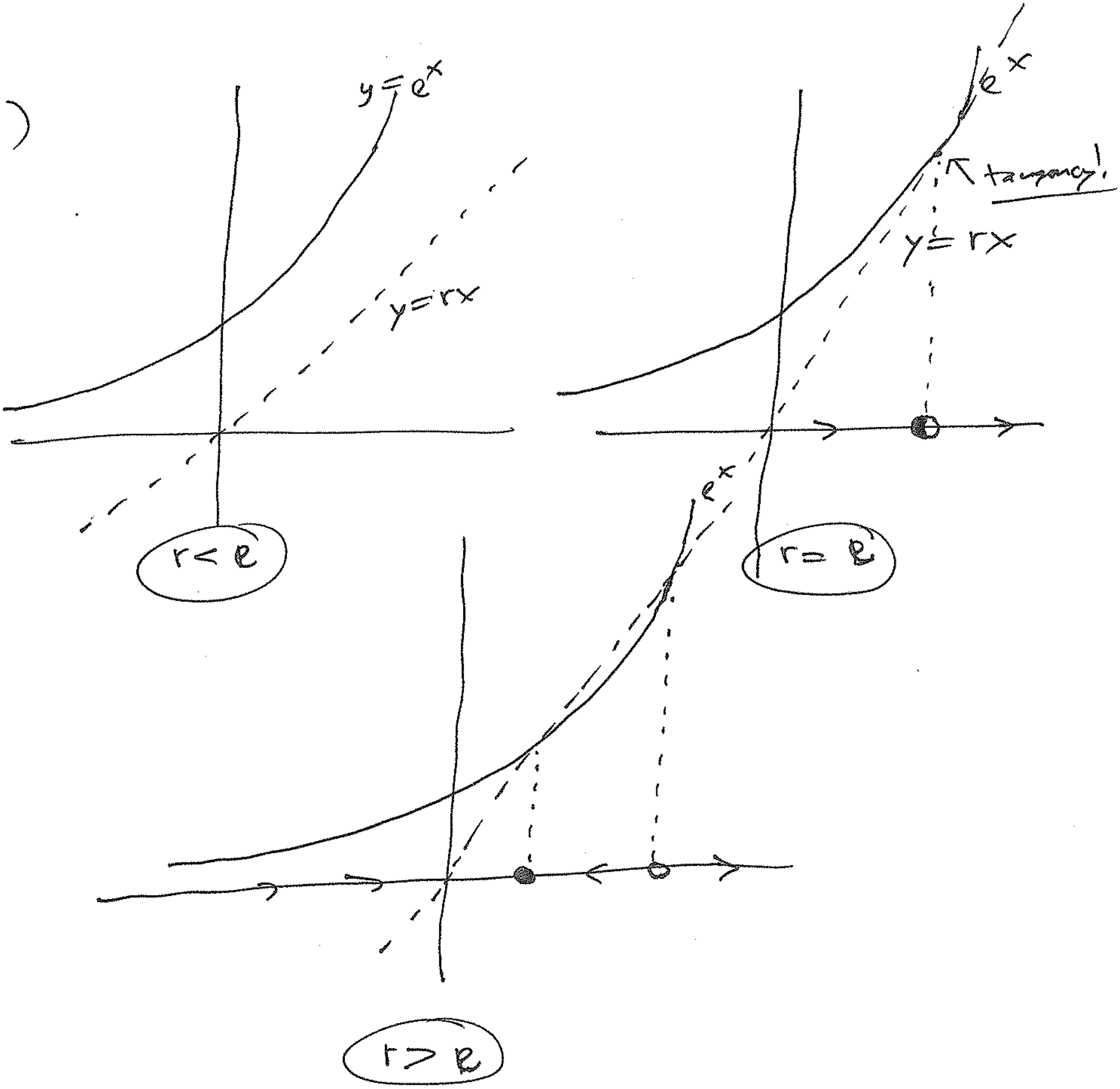
$\Longrightarrow r = 0$ or $x = 1$

$r = 0$ case : $r = e^x \Longrightarrow x = -\infty$! irrelevant

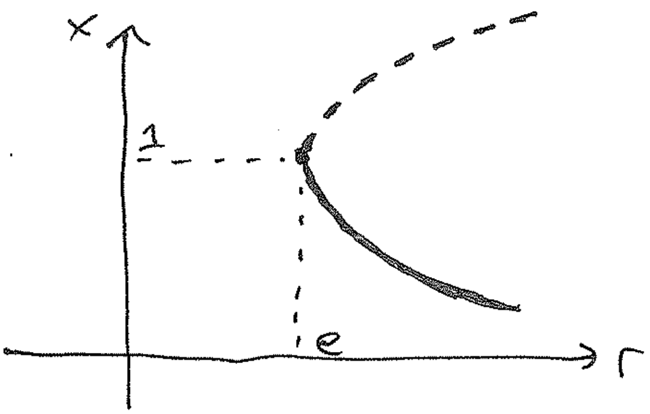
$x = 1$ case : $r = e^1 = e$

Thus, $(x_c, r_c) = (1, e)$

(b)



BIFURCATION DIAGRAM:



saddle-node bifurcation