Homework No.8, 550.391, Due November 7, 2011.

1. Strogatz, Problem 7.1.6. Specify the dimensions of all the quantities. Note that Strogatz' definition of $F(x)$ is not dimensionally correct! Please instead define the combination of μ times $F(x)$ by $\mu \cdot F(x) = C^{1/2} f(L^{-1/2} x)$. Finally, show that the system of equations in part b) is equivalent to the second-order equation

$$x'' + \mu F'(x) x' + x = 0.$$

Here $x' = dx/d\tau, \quad x'' = d^2x/d\tau^2$. In particular, conclude that for $F(x) = \frac{1}{3}x^3 - x$ the system is equivalent to van der Pol's equation.

2. Strogatz, Problem 7.1.9. [DOUBLE PROBLEM] In part (a) and (b), it is enough to derive the equations

$$\dot{R} = -k + \sin\phi, \quad R\dot{\phi} = \cos\phi - R.$$

It turns out that the dog does *not* catch the duck in finite time for $k = 1$, but it is non-trivial to show. In part (b) show mathematically that the dog catches the duck whenever $k > 1$. In part (c) you should find a fixed point of the system for (R, ϕ) when $k < 1$ and identify its type. Explain why this fixed point corresponds to a limit cycle for the dog's motion in the (x, y)-plane and describe this motion.

Hints: Let (x, y) be the coordinates of the dog in the Cartesian system with origin at the center of the pond and (ξ, η) be the coordinates of the dog in the Cartesian system with origin at the duck. You should be able to show that

$$R^2 = (x - \cos\theta)^2 + (y - \sin\theta)^2$$

$$\xi = R\cos\phi = 1 - x\cos\theta - y\sin\theta, \quad \eta = R\sin\phi = x\sin\theta - y\cos\theta$$

If you take time derivatives of R^2 and either $R\cos\phi$ or $R\sin\phi$, you will be able to derive the required equations, using $\dot{\theta} = 1$ and the stated result for $(\dot{x}, \dot{y})$. You should also be able to derive the following form of the system

$$\dot{\xi} = -k\frac{\xi}{\sqrt{\xi^2 + \eta^2}} + \eta, \quad \dot{\eta} = 1 - \xi - k\frac{\eta}{\sqrt{\xi^2 + \eta^2}}$$

in terms of the variables (ξ, η). Using `pplane8` or similar software for this form of the system will give you a good picture of how the dog moves from the duck's perspective! For part (b), derive an upper bound on $\dot{R}$. For part (c), use the result $x^2 + y^2 = (\xi - 1)^2 + \eta^2$ to identify the precise motion of the dog.

3. For each of the following two systems, determine whether it is a gradient system. If it is, then find V and sketch both the phase portrait and the equipotentials $V = const.$ For the latter, you may find it useful to use `ezcontour` in `MATLAB`.

(a) $\dot{x} = -y \sin x, \quad \dot{y} = -\cos x - \cos y$

(b) $\dot{x} = -y \cos x, \quad \dot{y} = -\sin x - \sin y$

4. Consider the system

$$\dot{x} = -x + \cos y, \quad \dot{y} = -2y + 2 \sin x.$$

(a) Show that this system is *not* of gradient type.

(b) Show that $V = \dot{x}^2 + \frac{1}{2}\dot{y}^2$ is a Lyapunov function for the system. Note that the fixed point of the system occurs at (x_*, y_*) defined by

$$x_* = \cos(\sin x_*) \doteq 0.768169156736796$$

$$y_* = \sin(\cos y_*) \doteq 0.694819690730788.$$

5. Strogatz 7.2.13.

6. Show using the Poincaré-Bendixson theorem that the system

$$\dot{x} = -x + y, \quad \dot{y} = -3x + 2y - y^3$$

has a periodic solution. *Hint:* Use as the outer boundary of the trapping region suitable segments of the lines $y = -4/3, x = -4/3, y = 2$, and the trajectory of the dynamics starting at $(2, 2)$.

(b) Using `MATLAB` or other numerical software, plot the limit cycle of part (a) and verify that it lies in the trapping region you constructed.