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AMS 550.391 - Dynamical Systems
Homework #4
10/14/2011

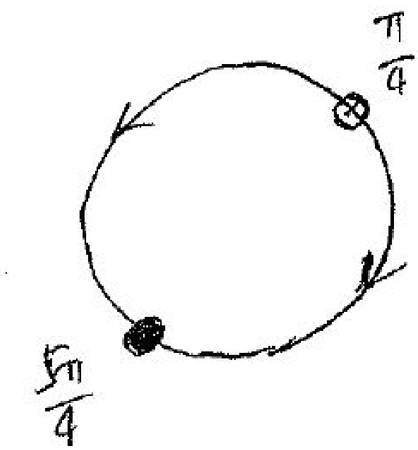
1. Find and classify all of the fixed points of the system

$$\dot{\theta} = \sin(\theta) - \cos(\theta).$$

Sketch the phase portrait on the circle.

Root condition implies (note we only need to find the zeros between 0 and 2π),

$$\sin\theta - \cos\theta = 0 \quad \Rightarrow \quad \theta = \frac{\pi}{4}, \frac{5\pi}{4}.$$



Hence $\frac{\pi}{4}$ is unstable and $\frac{5\pi}{4}$ is stable. ■

2. Double Points (a) Consider the vector field on the circle given by $\dot{\theta} = \sin(\theta)$. Show that this system has a single-valued potential $V(\theta)$, i.e. for each point on the circle, there is a well-defined value of V such that $\dot{\theta} = -dV/d\theta$. (As usual, θ and $\theta + 2\pi k$ are to be regarded as the same point on the circle, for each integer k.)
(b) Now consider $\dot{\theta} = 1$. Show that there is no single-valued potential $V(\theta)$ for this vector field on the circle.
c) What's the general rule? When does $\dot{\theta} = f(\theta)$ have a single-valued potential?
d) In Homework # 2, Problem 4, you were asked to give an analytical proof that periodic solutions are impossible for vector fields on the line. Explain why your arguments there *do not* carry over to vector fields on the circle. Specifically, what part of the argument fails?

a) We can find the potential,

$$V'(\theta) = -\sin\theta \quad \Rightarrow \quad V(\theta) = \cos\theta + C.$$

Since $V(\theta)$ has period 2π, it is single-valued on the circle.

b) Again we find the potential

$$V'(\theta) = -1 \quad \Rightarrow \quad V(\theta) = -\theta + C,$$

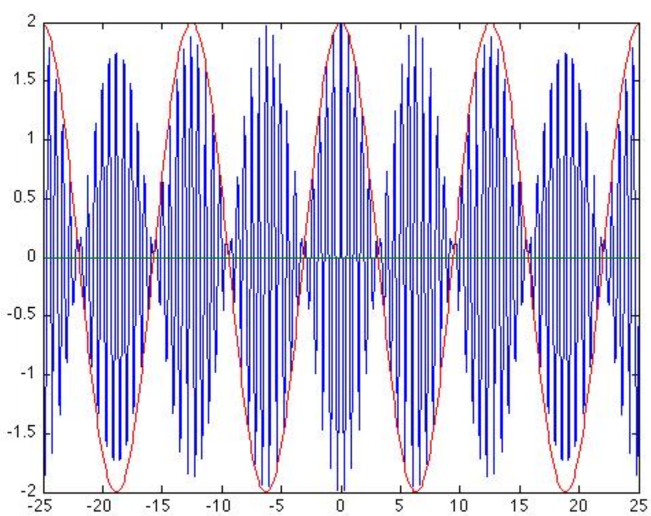
which changes by 2π every time θ goes around the circle and thus is not single-valued.

c) $-V'(\theta) = f(\theta)$. We can choose certain constant such that $V(\theta) = -\int_0^\theta f(\phi)d\phi$. $V(\theta)$ is single-valued on the circle when $V(0) = V(2\pi)$ and $V(2\pi) - V(0) = \int_0^{2\pi} -f(\phi)d\phi$. So $\int_0^{2\pi} f(\phi)d\phi = 0$ gives a general rule.

d) Part b) gives an example of periodic solution. The reason that the result and the arguments on the line don't carry over is because the potential here may not be single-valued. Hence $V(x(t_2)) = V(x(t_1))$ does not imply $V'(x) \equiv 0$ on the x-interval $x([t_1, t_2])$ and the arguments break down.
However, if the potential is assumed to be single-valued, we can prove periodic solution doesn't exist. Let's prove by contradiction. Suppose there is periodic solution. In that case there can be no fixed points. So without loss of generality we can assume $f(\theta) > 0$, then the average of $f(\theta)$ over all $0 < \theta < 2\pi$ is also > 0. That contradicts the general rule in part c). ■

3. Graph $x(t) = \cos(10t) + \cos(11t)$ for $-25 < t < 25$. You should find that the amplitude of the oscillations is *modulated*—it grows and decays periodically.
a) What is the period of the amplitude modulations?
b) Solve this problem analytically, using a trigonometric identity that converts sums of sines and cosines to products of sines and cosines.

a) The period of amplitude is $\frac{2\pi}{11-10} = 2\pi$.



b) Use trigonometric identity,

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}.$$

We have

$$x(t) = \cos(10t) + \cos(11t) = 2\cos(\frac{21}{2}t)\cos(\frac{t}{2}).$$

In the figure we have plotted the "envelope function" $2\cos(t/2)$ which describes the slow modulation of the amplitude. Note that the amplitude $A = 2|\cos(t/2)|$ is indeed a periodic function with period 2π. ■

4. Strogatz, Problem 4.3.2. The oscillation period for the nonuniform oscillator is given by the integral

$$T = \int_{-\pi}^{\pi} \frac{d\theta}{\omega - a\sin\theta},$$

where $\omega > a > 0$. Evaluate this integral.

(a) $u = \tan\frac{\theta}{2} \Rightarrow \theta = 2\arctan u$. So $d\theta = \frac{2du}{1+u^2}$.
(b)

$$\frac{2u}{1+u^2} = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{2\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}} = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\theta$$

(c) As $\theta \to \pm\pi$, $\tan\frac{\theta}{2} \to \pm\infty$. So we can rewrite the limits with respect to u to be $\int_{-\infty}^{+\infty}$.
(d)

$$T = \int_{-\infty}^{+\infty} \frac{\frac{2du}{1+u^2}}{\omega - \frac{2au}{1+u^2}} = \int_{-\infty}^{+\infty} \frac{2du}{\omega(1+u^2) - 2a\omega}$$

(e) Let $\lambda = \frac{a}{\omega}$,

$$\begin{aligned}
T =& \frac{2}{\omega}\int_{-\infty}^{+\infty} \frac{du}{1+u^2-2\lambda u} \\
=& \frac{2}{\omega}\int_{-\infty}^{+\infty} \frac{du}{(1-\lambda^2)+(u-\lambda)^2} \\
=& \frac{2}{\omega}\int_{-\infty}^{+\infty} \frac{dv}{(1-\lambda^2)+v^2}, \quad \text{let } v = u - \lambda \\
=& \frac{2}{\omega}\frac{1}{\sqrt{1-\lambda^2}}\arctan(\frac{u}{\sqrt{(1-\lambda^2)}})|_{-\infty}^{+\infty} \\
=& \frac{2\pi}{\omega\sqrt{1-\lambda^2}} \\
=& \frac{2\pi}{\sqrt{\omega^2-a^2}}.
\end{aligned}$$

Remark: We can also evaluate the integral using complex analysis, namely Cauchy integral formula. Let's look at a simplified equivalent form

$$I = \int_0^{2\pi} \frac{d\theta}{\gamma - \sin\theta}.$$

Use change of variable $z = e^{i\theta} \in \mathcal{S}^1$ (unit circle centered at the origin in complex plane), we can get

$$dz = ie^{i\theta}d\theta \;\; \Rightarrow \;\; \frac{dz}{iz} = d\theta;$$
$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}.$$

Plug into I, we have

$$\begin{aligned} I = &\oint_{\mathcal{S}^1} \frac{\frac{dz}{iz}}{\gamma - \frac{z - z^{-1}}{2i}} \\ =&2\oint_{\mathcal{S}^1} \frac{dz}{z(2i\gamma - z + z^{-1})} \\ =&2\oint_{\mathcal{S}^1} \frac{dz}{1 + (2i\gamma)z - z^2} \end{aligned}$$

The singular points of the integrand correspond to the zeros of $z^2 - (2i\gamma)z - 1$. When $\gamma > 1$, there are two zeros

$$z_\pm = \frac{2i\gamma \pm \sqrt{4 - 4\gamma^2}}{2} = i(\gamma \pm \sqrt{\gamma^2 - 1})$$

For $\gamma > 1$,

$$-iz_+ = \gamma + \sqrt{\gamma^2 - 1} > 1 \;\; \Rightarrow \;\; |z_+| > 1$$

On the other hand,

$$\begin{aligned} 2 < 2\gamma \;\; &\Rightarrow \;\; \gamma^2 - 2\gamma + 1 < \gamma^2 - 1 \\ &\Rightarrow \;\; \gamma - 1 < \sqrt{\gamma^2 - 1} \\ &\Rightarrow \;\; \gamma - \sqrt{\gamma^2 - 1} < 1. \end{aligned}$$

Hence, we only have one singular point inside $\mathcal{S}^1$. Use Cauchy integral formula, we have

$$\begin{aligned} I =& -4\pi i \frac{1}{2\pi i} \oint_{\mathcal{S}^1} \frac{dz}{(z - z_+)(z - z_-)} \\ =& -4\pi i \frac{1}{z - z_+}\Big|_{z = z_-} \\ =& -4\pi i \frac{1}{-2i\sqrt{\gamma^2 - 1}} = \frac{2\pi}{\sqrt{\gamma^2 - 1}} \end{aligned}$$

■

5. Draw the phase portrait of the system

$$\dot{\theta} = \mu \sin(\theta) - \sin(3\theta)$$

as a function of the control parameter μ. Classify the bifurcations that occur as μ varies, and find all the bifurcation values of μ.

The root condition implies

$$\mu \sin\theta = \sin(3\theta) = 3\sin\theta\cos^2\theta - \sin^3\theta \quad \Rightarrow \quad \mu = 3\cos^2\theta - \sin^2\theta \text{ or } \sin\theta = 0.$$

The tangency condition implies

$$\mu\cos\theta = 3\cos(3\theta) = 3(\cos^3\theta - 3\cos\theta\sin^2\theta) \quad \Rightarrow \quad \mu = 3(\cos^2 - 3\sin^2\theta) \text{ or } \cos\theta = 0.$$

If $\sin\theta = 0$, then $\cos\theta \neq 0$ and we have

$$\mu_c = 3(\cos^2\theta - 3\sin^2\theta) = 3.$$

If $\cos\theta = 0$, then $\sin\theta \neq 0$ and we have

$$\mu_c = 3\cos^2\theta - \sin^2\theta = -1.$$

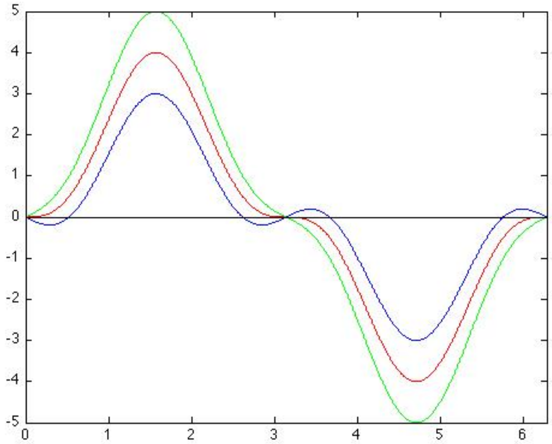
If neither $\sin\theta$ nor $\cos\theta$ is zero, we have

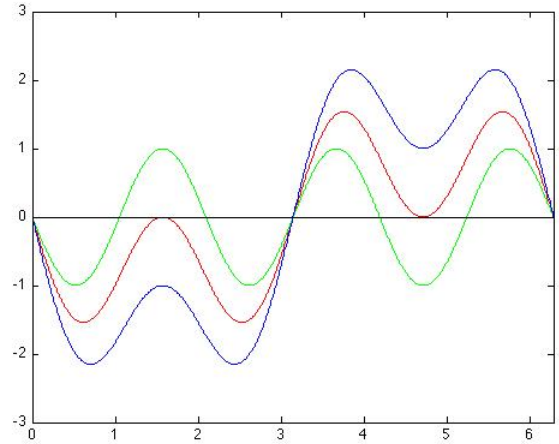
$$3\cos^2\theta - \sin^2\theta = 3(\cos^2 - 3\sin^2\theta) \quad \Rightarrow \quad \sin\theta = 0.$$

Therefore, the critical points are $(\theta_c, \mu_c) = (0, 3), (\pi, 3), (\frac{\pi}{2}, -1), (\frac{3\pi}{2}, -1)$, where $k \in \mathbb{Z}$. We can draw the phase portrait using the following `MATLAB` code

```
th=0:0.01:2*pi;
plot(th,3*sin(th)-sin(3*th),'-r',th,4*sin(th)-sin(3*th),'-g',...
     th,2*sin(th)-sin(3*th),'-b',th,0*th,'-k')
axis([0 2*pi -5 5])

figure
plot(th,-1*sin(th)-sin(3*th),'-r',th,0*sin(th)-sin(3*th),'-g',...
     th,-2*sin(th)-sin(3*th),'-b',th,0*th,'-k')
axis([0 2*pi -3 3])
```





Here the left figure plots $f(\theta, \mu)$ for values of μ near 3 and the right one plots $f(\theta, \mu)$ for values of μ near -1.
From the left graph, we can see that when $\mu < 3$, the system has two unstable fixed points near 0 (while 0 is a stable fixed point) and two stable fixed points near π (while π is an unstable fixed point); when $\mu > 3$, the system has one unstable fixed point at 0 and one stable fixed point at π. Therefore we have a subcritical pitchfork bifurcation at critical point $(0, 3)$ and a supercritical pitchfork bifurcation at critical point $(\pi, 3)$.
From the right graph, we can see that the system has two fixed points near $\frac{\pi}{2}$ when $\mu > -1$ and they disappear when $\mu < -1$; and near $\frac{3\pi}{2}$, the system has two fixed points when $\mu > -1$ and they disappear when $\mu < -1$. Therefore we have a saddle node bifurcation both at $(\frac{\pi}{2}, -1)$ and $(\frac{3\pi}{2}, -1)$. ■

6. Strogatz, Problem 4.5.1

(a) Graph $f(\phi)$.

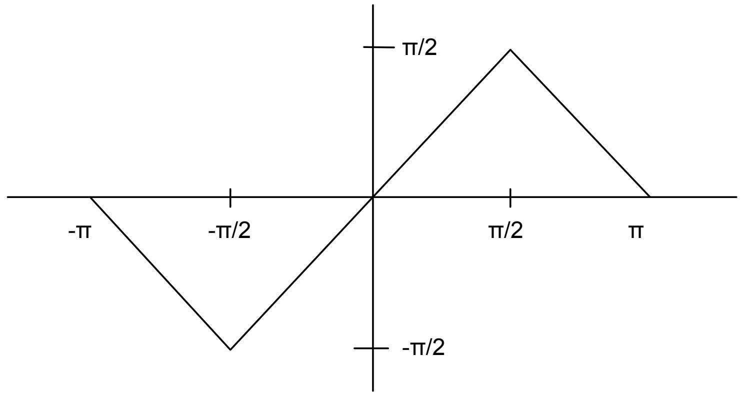
(b) Find the range of entrainment.

(c) Assuming that the firefly is phase-locked to the stimulus, find a formula for the phase difference ϕ^*.

(d) Find a formula for $\dot{\theta}$.

(e) Why does the root scaling law not apply near the bifurcation?

a)



b) Let $y = \Omega - \omega$ intersects $y = Af(\phi)$, $|f(\phi)| \leq \frac{\pi}{2}$. Then$|\omega - \Omega| \leq \frac{\pi}{2}A$. Therefore, the range of entrainment is $[\omega - \frac{\pi}{2}A, \omega + \frac{\pi}{2}A]$.

c) Set $\Omega - \omega - Af(\phi) = 0$, so $f(\phi^*) = \frac{\Omega - \omega}{A}$. From the graph of $f(\phi)$ we know that the only stable equilibrium is $\phi^* = \frac{\Omega-\omega}{A}$.

d) Let $\mu = \frac{\Omega-\omega}{A}$,

$$\begin{aligned}
\dot{\theta} &= \int_{-\pi/2}^{3\pi/2} \frac{d\phi}{\Omega - \omega - Af(\phi)} \\
&= \frac{1}{A} \int_{-\pi/2}^{3\pi/2} \frac{d\phi}{\mu - f(\phi)} \\
&= \frac{1}{A} \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\mu - \phi} + \int_{\pi/2}^{3\pi/2} \frac{d\phi}{(\mu - \pi) + \phi} \\
&= \frac{1}{A} \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\mu - \phi} + \int_{-\pi/2}^{\pi/2} \frac{d\phi}{\mu - \phi} \\
&= -\frac{2}{A} \ln(\mu - \phi)\Big|_{-\pi/2}^{\pi/2} \\
&= \frac{2}{A} \ln\left(\frac{\mu + \frac{\pi}{2}}{\mu - \frac{\pi}{2}}\right)
\end{aligned}$$

Then as $\mu \to \frac{\pi}{2}^+ \Rightarrow \dot{\theta} \to \infty$

e) The reason is that $f(\phi)$ is only piecewise linear, the system is not smooth at $\frac{\pi}{2}$. Therefore the root scaling law doesn't apply. Statements about the "generic" scaling behavior at a saddle-node bifurcation always assume that the vector field is very smooth. ■