Homework No.2, 550.391, Due Septemember 17, 2010.

1. Explain this paradox: the driven system $\ddot{x} = -\cos t$ has an exact solution $x(t) = \cos t$ which oscillates along the one-dimensional x-axis. But Strogatz claimed that one-dimensional systems cannot oscillate!

2. Plot the potential $V(x)$ and identify all of the equilibrium points and their stability for the vector field on the positive x-axis $x > 0$:

$$\dot{x} = \frac{\ln x - \ln^3 x}{x}.$$

3. Repeat problem 2 for the vector field

$$\dot{x} = \cos x - \cos^3 x$$

on the real axis $-\infty < x < +\infty$.

4. This problem develops another proof of absence of periodic solutions for 1-dimensional dynamical systems. It is based on the result established by Strogatz for potentials V in Section 2.7:

$$\frac{dV}{dt} = -\left(\frac{dV}{dx}\right)^2.$$

For any solution $x(t)$ of a 1-dimensional ODE, use the above result to show that:
(a) For $t_2 > t_1$, $V(x(t_2)) \leq V(x(t_1))$.
(b) For $t_2 > t_1$, $V(x(t_2)) = V(x(t_1))$ if and only if $V'(x) \equiv 0$ on the x-interval $x([t_1, t_2])$ = all values taken on by the solution $x(t)$ in the time interval $[t_1, t_2]$.
(c) Use (b) to show that the only solutions with period $T > 0$ are equilibria $x(t) = x_*$, $V'(x_*) = 0$. (*Note:* Equilibrium solutions are periodic with *any* period $T > 0$!)

5. Strogatz, Problem 2.8.3, but with initial-value problem $\dot{x} = 1 + x^2$, $x(0) = 0$ instead (and everything else the same). Note there is a typo in Strogatz' statement of the last part of the problem, where he intended to ask for a plot of $\ln E$ vs. $\ln(\Delta t)$.

6. Strogatz, Problem 2.8.4, but with initial-value problem $\dot{x} = 1 + x^2$, $x(0) = 0$ instead (and everything else the same). See note in Problem 5 above.

7. Strogatz, Problem 2.8.5, but with initial-value problem $\dot{x} = 1 + x^2$, $x(0) = 0$ instead (and everything else the same). See note in Problem 5 above.