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AMS 550.391 - Dynamical Systems
Homework #2
9/16/2011

1. Explain this paradox: the driven system $\ddot{x} = -\cos t$ has an exact solution $x(t) = \cos t$ which oscillates along the one-dimensional x-axis. But Strogatz claimed that one-dimensional systems cannot oscillate!

By one-dimensional system Strogatz meant one-dimensional first-order autonomous system which has the form $\dot{x} = f(x)$. This problem is, however, nonautonomous because $-\cos t$ depends on the time t; also the equation involves higher-order derivative. Hence $\ddot{x} = -\cos t$ is really a one-dimensional second-order nonautonomous equation; or introduce $x_1 = x, x_2 = \dot{x}, x_3 = t$, we can convert it into a first order autonomous system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\cos x_3, \\ \dot{x}_3 = 1 \end{cases}$$

The original ODE is three-dimensional. ∎

2. Plot the potential $V(x)$ and identify all of the equilibrium points and their stability for the vector field on the positive x-axis $x > 0$:

$$\dot{x} = \frac{\ln x - \ln^3 x}{x}.$$

Let

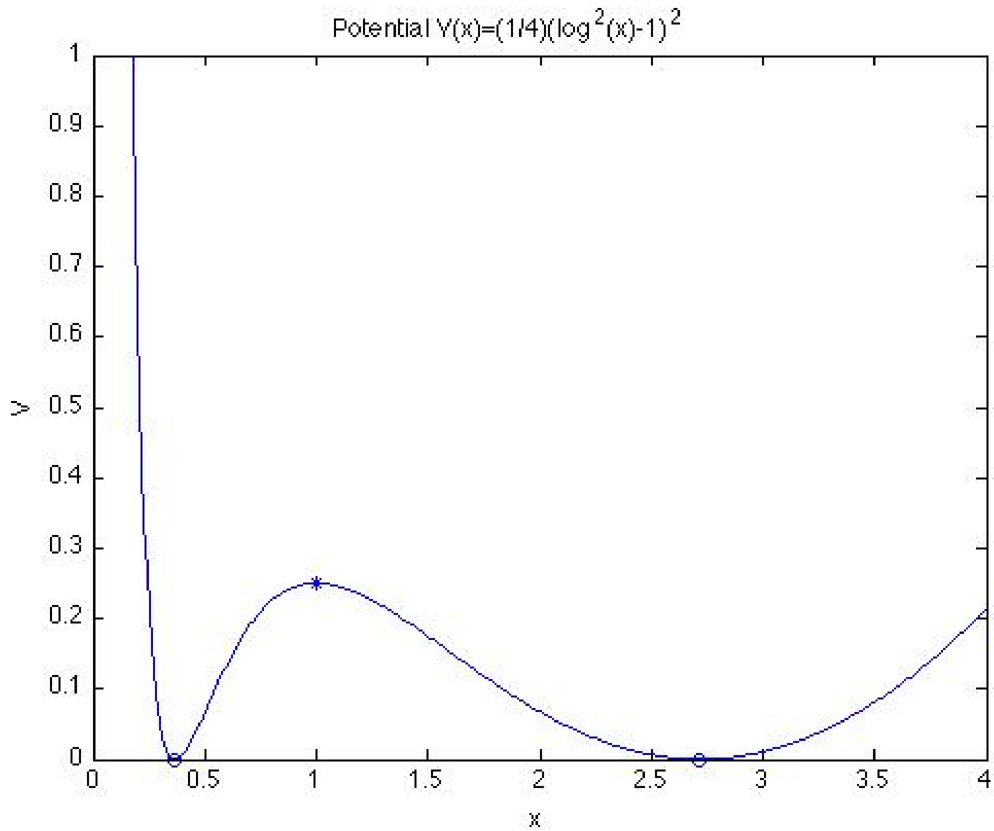
$$f(x) := \frac{\ln x - \ln^3 x}{x}.$$

Then the potential is given by

$$V(x) = -\int \frac{\ln x - \ln^3 x}{x} dx = -\int (\ln x - \ln^3 x) d(\ln x) = \frac{1}{4}\ln^4 x - \frac{1}{2}\ln^2 x + C.$$

We can choose $C = \frac{1}{4}$ so that $V(x) = \frac{1}{4}(\ln^2 x - 1)^2$.
To find the fixed points, we solve $f(x) = 0$ for $\ln x = 0, \pm 1$. Hence the fixed points are $x = 1, e, e^{-1}$. We can plot the potential using the following `MATLAB` code

```
V=inline('((log(x)).^2-1).^2/4','x');
xx=0.1:0.01:4;
plot(xx,V(xx))
axis([0 4 0 1])
hold on
plot(1/exp(1),V(1/exp(1)),'bo',1,V(1),'b*',exp(1),V(exp(1)),'bo')
xlabel('x');ylabel('V');title('Potential V(x)=(1/4)(log^2(x)-1)^2')
```



From the graph of potential, we know $x = e, e^{-1}$ are stable and $x = 1$ is unstable. ∎

3. Repeat problem 2 for the vector field

$$\dot{x} = \cos x - \cos^3 x$$

on the real axis $-\infty < x < +\infty$.

Let

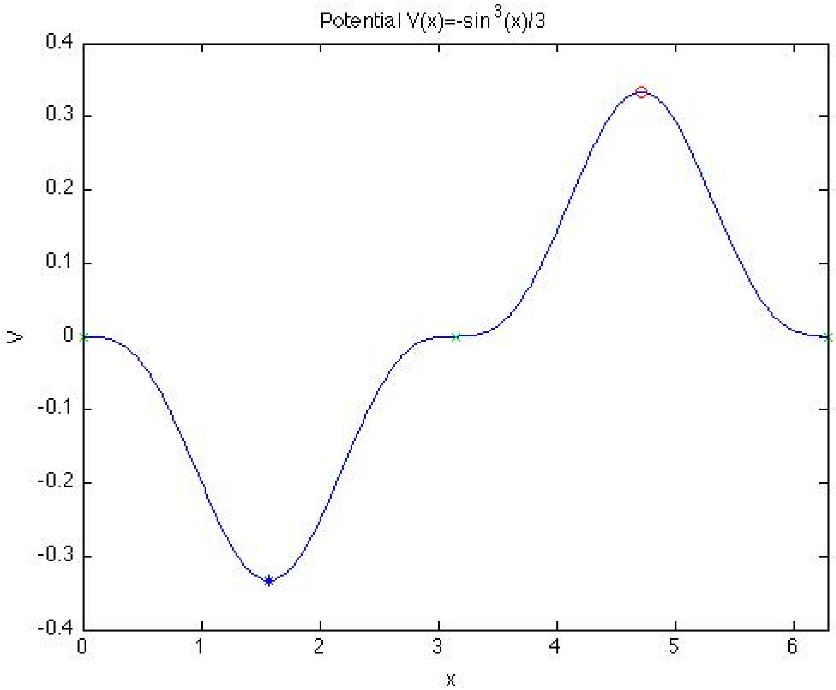
$$f(x) := \cos x - \cos^3 x = \cos x(1 - \cos^2 x) = \cos x \sin^2 x.$$

Then the potential is given by

$$V(x) = -\int \cos x \sin^2 x dx = -\int \sin^2 x d(\sin x) = -\frac{\sin^3 x}{3}.$$

To find the fixed points, we solve $f(x) = 0$ for $\cos x = 0, \pm 1$. Hence the fixed points are $x = \frac{k\pi}{2}$, where $k \in \mathbb{Z}$. We can plot the potential using the following MATLAB code

```
V=inline('-(sin(x))^3/3','x');fplot(@(x) V(x), [0 2*pi])
hold on
plot(pi/2,V(pi/2),'b*',3*pi/2,V(3*pi/2),'ro',0,V(0),'gx',pi,V(pi),'gx',2*pi,V(2*pi),'gx')
xlabel('x');ylabel('V');title('Potential V(x)=-sin^3(x)/3')
```



Note

$$V(x) = \begin{cases} 1 & x = 2k\pi + \frac{3\pi}{2} \\ 0 & x = k\pi \\ -1 & x = 2k\pi + \frac{\pi}{2} \end{cases}$$

From the graph of the potential, we know that $x = k\pi$ are semi-stable fixed points, $x = 2k\pi + \frac{\pi}{2}$ are stable and $x = 2k\pi + \frac{3\pi}{2}$ are unstable. ■

4. This problem develops another proof of absence of periodic solutions for 1-dimensional dynamical systems. It is based on the result established by Strogatz for potentials V in Section 2.7:

$$\frac{dV}{dt} = -\left(\frac{dV}{dx}\right)^2.$$

For any solution $x(t)$ of a 1-dimensional ODE, use the above result to show that:
(a) For $t_2 > t_1$, $V(x(t_2)) \leq V(x(t_1))$.
(b) For $t_2 > t_1$, $V(x(t_2)) = V(x(t_1))$ if and only if $V'(x) \equiv 0$ on the x-interval $x([t_1, t_2]) =$ all values taken on by the solution $x(t)$ in the time interval $[t_1, t_2]$.
(c) Use (b) to show that the only solutions with period $T > 0$ are equilibria $x(t) = x_*$, $V'(x_*) = 0$. (*Note:* Equilibrium solutions are periodic with *any* period $T > 0$!)

(a) On the one hand,

$$\int_{t_1}^{t_2} \frac{d}{dt} V(x(t))\, dt = \int_{x(t_1)}^{x(t_2)} dV(x(t)) = V(x(t_2)) - V(x(t_1))$$

where the first equality follows from the chain rule, and the second from the Fundamental Theorem of Calculus. On the other hand,

$$\int_{t_1}^{t_2} \frac{d}{dt} V(x(t))\, dt = -\int_{t_1}^{t_2} \left(\frac{dV(x(t))}{dx} \right)^2 dt \leq 0 \tag{1}$$

The inequality is true because the integrand is always ≥ 0. Therefore,

$$V(x(t_2)) - V(x(t_1)) \leq 0 \Rightarrow V(x(t_2)) \leq V(x(t_1))$$

(b) First, if $V'(x) \equiv 0$ on $x([t_1, t_2])$, then clearly $V(x(t_2)) = V(x(t_1))$. Conversely, if $V(x(t_2)) = V(x(t_1))$, then $V(x(t_2)) - V(x(t_1)) = 0$. But from Equation (1), that integral vanishes only if the integrand vanishes identically on the entire interval. So $V'(x) = 0$ on $x([t_1, t_2])$.

(c) Assume that $x(t)$ is periodic with period T. Then $V(x(t))$ is also periodic with period T. Then $V(x(T)) = V(x(0))$ and we can use the result in part (b) to know $V'(x) = 0$ identically on $x([0, T])$. Hence $x(t) = x_*$ for $t \in [0, T]$. But since $x(t)$ is periodic, $x(t) = x_*$ for all t.

■

5-7. Strogatz, Problem 2.8.3, 2.8.4, 2.8.5 The goal is to test the Euler, Heun and Runge-Kutta method on the initial value problem $\dot{x} = 1 + x^2, \;\; x(0) = 0$.
a) Solve the problem analytically. What is the exact value of $x(1)$?
b) Using the Euler, Heun and Runge-Kutta method with step size $\Delta t = 1$, estimate $x(1)$ numerically–call the result $\hat{x}(1)$. Then repeat, using $\Delta t = 10^{-n}$, for $n = 1, 2, 3, 4$.
c) Plot the error $E = |\hat{x}(1) - x(1)|$ as a function of Δt. Then plot $\ln E$ vs. $\ln(\Delta t)$. Explain the result.

a) From separation of variables, we have

$$\int \frac{dx}{1+x^2} = \int dt \;\; \Rightarrow \;\; \arctan x = t + C \;\; \Rightarrow \;\; x(t) = \tan(t + C).$$

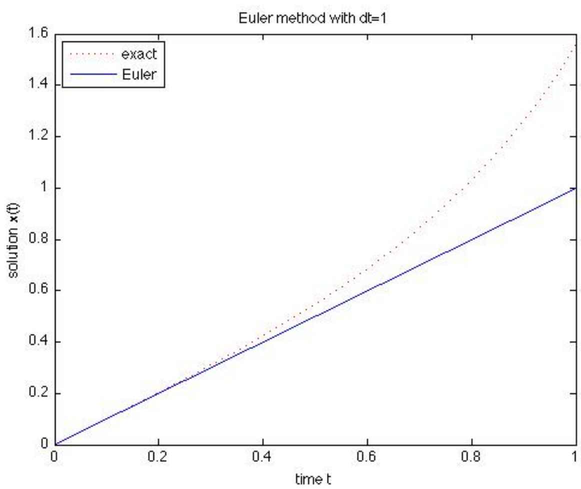
Plug in the initial condition $x(0) = 0$, we have $C = 0$. Therefore $x(t) = \tan t$ and the exact value of $x(1) = \tan 1$.
b) We first put the equation into the `f_RHS.m`

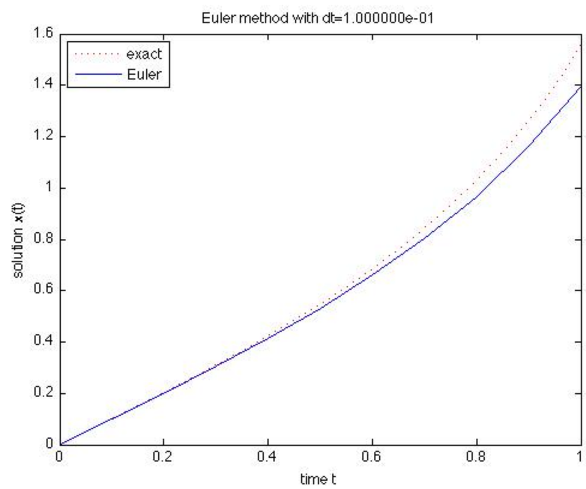
```
function [dydt] = f_RHS( y, t )
dydt(1) = 1+y(1)^2;
```

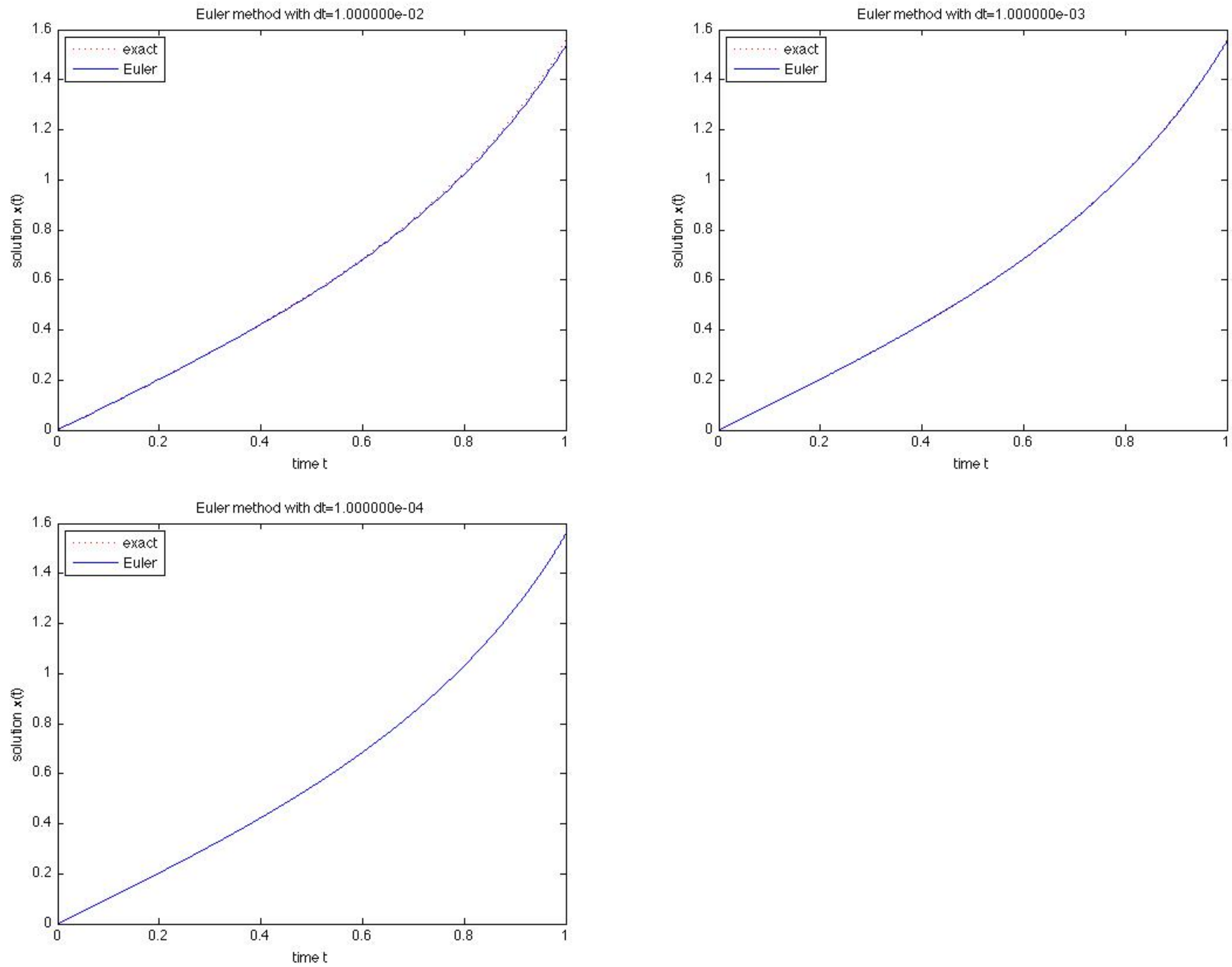
Then we use `ode.m` and the following `MATLAB` code

```
Y=inline('tan(x)');
ye=Y(1);
tt=0:1e-4:1;

for ii=0:4

Ns=10^ii;
[y,t]=ode(1,0,0,1,Ns,1,0);
figure; plot(tt,Y(tt),'r:',t,y,'b-')
xlabel('time t'); ylabel('solution x(t)');
legend('exact','Euler','Location','NorthWest')
title(sprintf('Euler method with dt=%d',1/Ns))
ya=y(Ns+1);
err(ii+1)=abs(ya-ye);
dt(ii+1)=1/Ns;

end
```

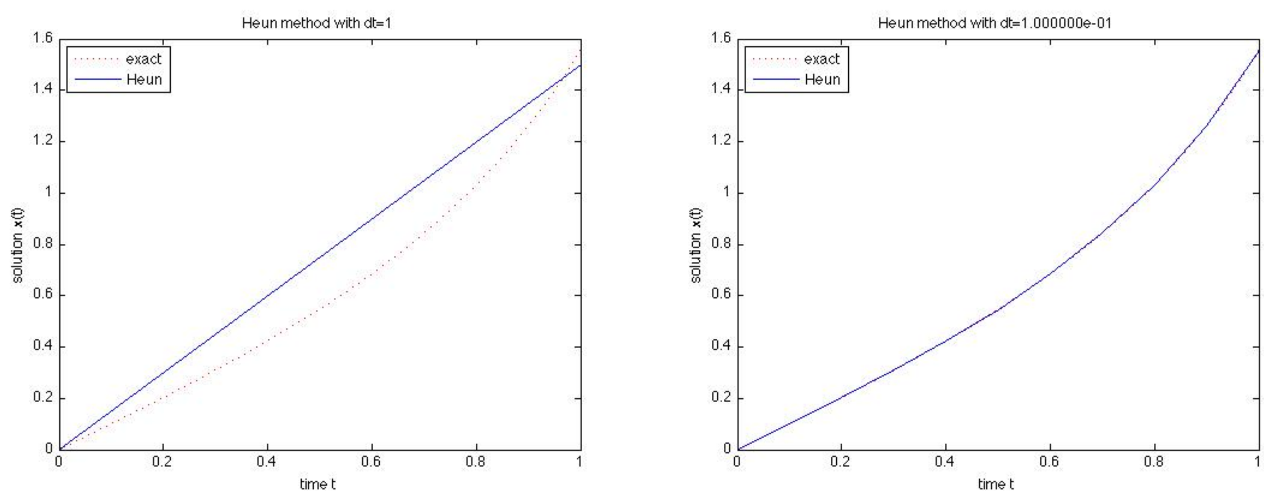
The plots are as follow

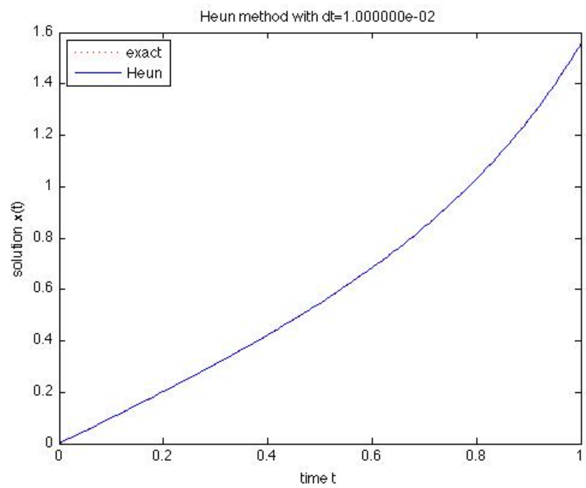


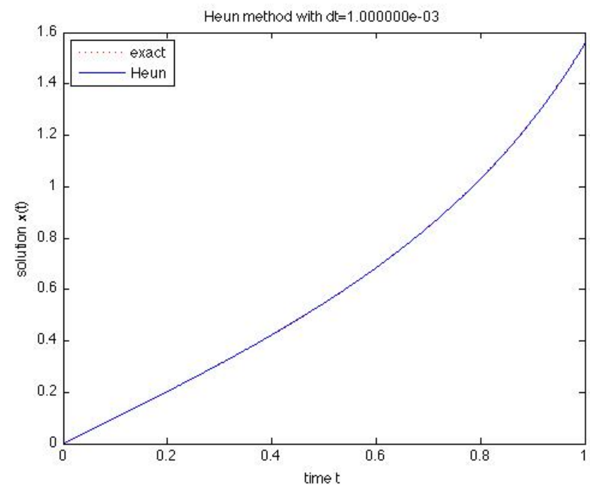


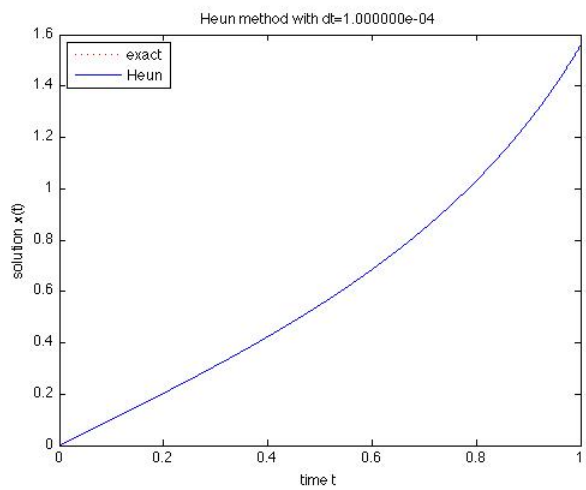


For the Heun method, we just need to slightly change the above MATLAB code, which would give us the following result

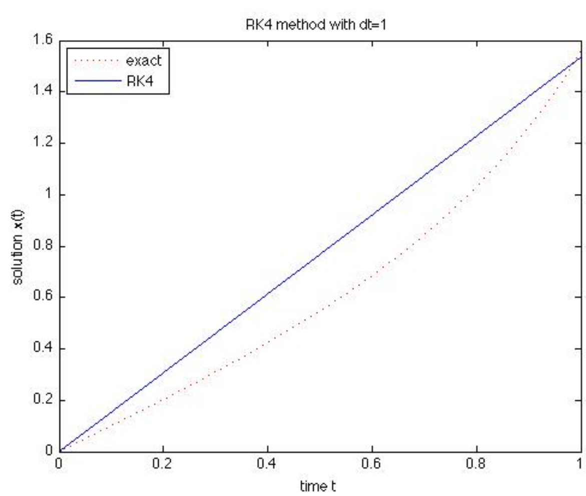


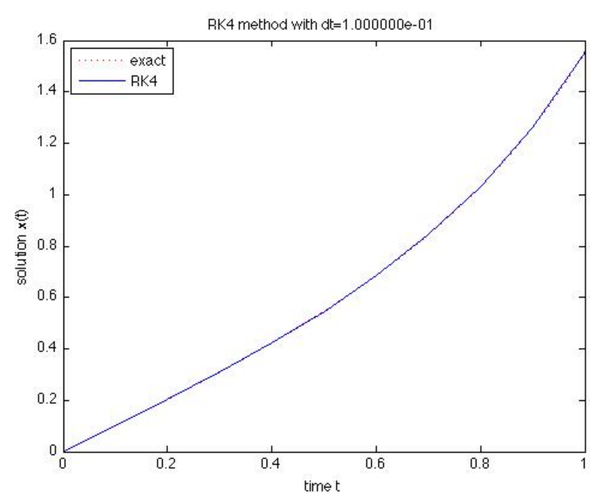


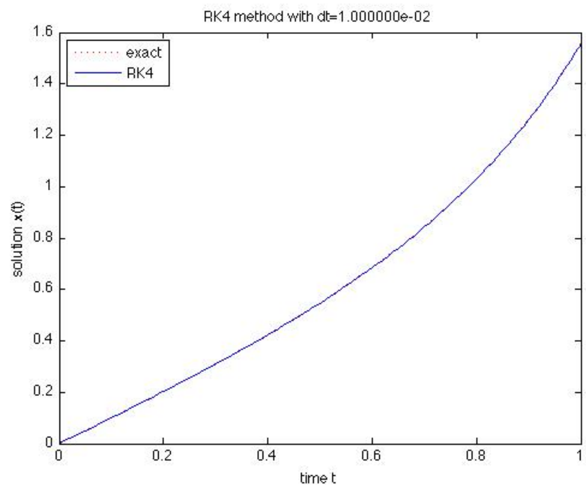


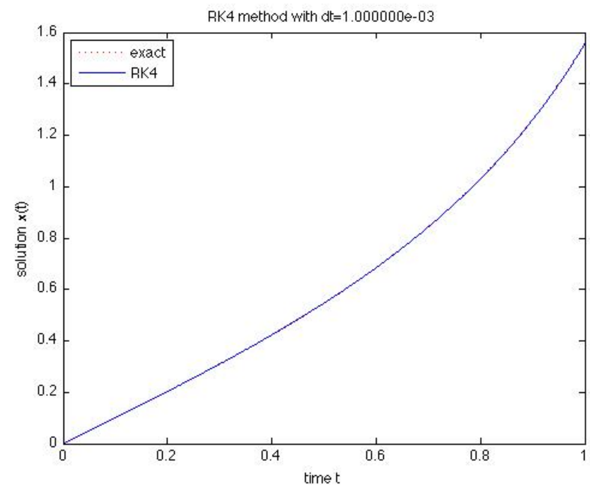


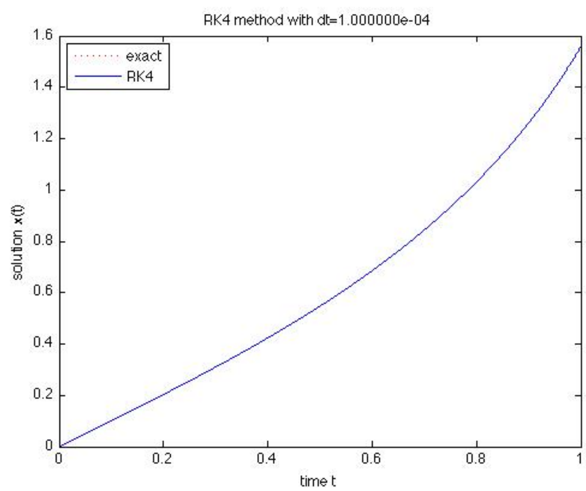
For the Runge-Kutta method, we have the following plots





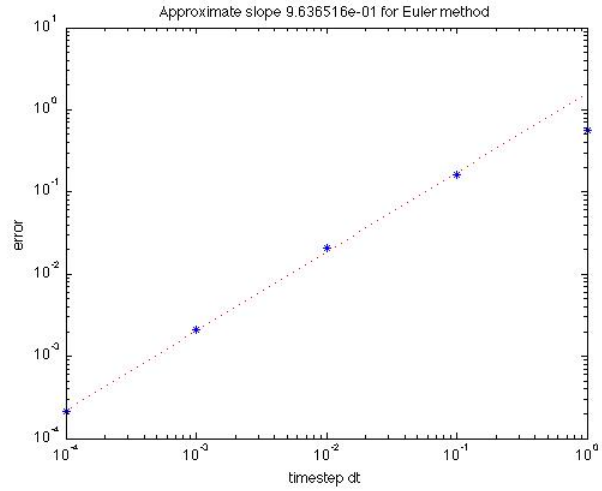


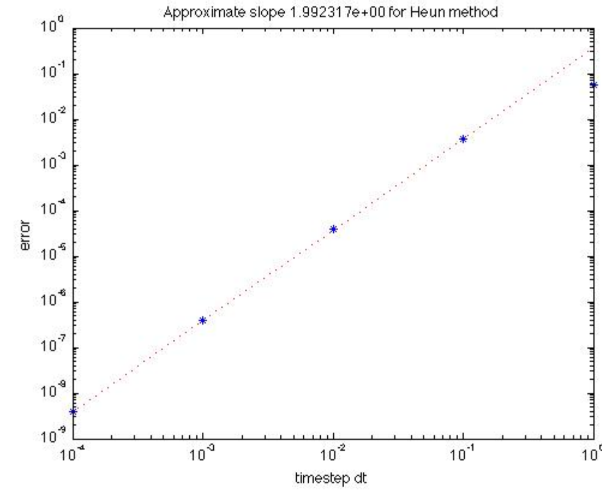


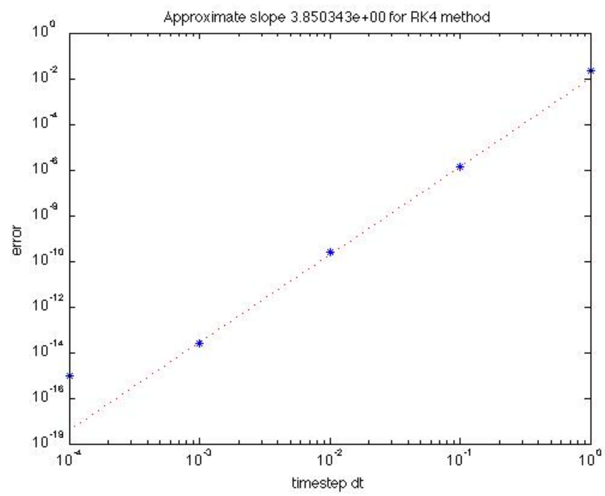


c) We use the following `MATLAB` code to generate the plot

```
figure
loglog(dt,err,'b*')
pp=polyfit(log(dt(2:5)),log(err(2:5)),1);
hold on; loglog(dt,exp(polyval(pp,log(dt))),'r:')
title(sprintf('Approximate slope %d for Euler method',pp(1)))
xlabel('timestep dt'); ylabel('error');
```







We can make sense of the plots by remembering that for an order n method the error $E \propto dt^n$, so $\ln E \approx n \ln dt$. As $\ln dt$ decreases linearly, the $\ln E$ decreases linearly proportional to the order of the method. So the Heun method decreases twice as fast as the ordinary Euler method, and the Runge-Kutta method approximately four times as fast.

Note that for Euler and Heun methods the error for $dt = 1$ was not used in calculating the approximate order of the method, because $dt = 1$ is not small enough to see the asymptotic scaling of the error as dt^p. This amounts to ignoring the first point on the far right.

On the other hand, for the 4th-order Runge-Kutta method, the error for $dt = 10^{-4}$ was not used, which amounts to ignoring the last point on the far left. The reason is that now the error in the Runge-Kutta method is of order 10^{-16}, which is the same size as the precision of the arithmetic in matlab (16 decimal places). This explains why one no longer sees error $\propto dt^4$ for that last point.

■